3a)

5/5 a) $\sigma_i = \pm 1$, $i = 1, 2, \ldots, N$

$$\tau_i = \sigma_i \sigma_{i+1} = (\pm 1)(\pm 1) = \pm 1 \ \#$$

$\sigma_1 = 1, -1$ $\quad \tau_1 = \sigma_1 \sigma_2 = (1,-1)(1,-1)$

$\sigma_2 = 1, -1$ $\quad = (1\cdot 1, -1(-1), 1(-1), -1(1))$

$\sigma_3 = 1, -1$ $\quad = (1, 1, -1, -1)$

similarly

$\tau_2 = (1, 1, -1, -1)$

$\tau_3 = (1, 1, -1, -1)$, τ_i's range over the values of σ_i twice!

b) $$Z = \sum_{\sigma_1} \cdots \sum_{\sigma_n} \exp\left(y \sum_{i=1}^{N-1} \sigma_i \sigma_{i+1}\right), \quad y = \beta J$$

$$= 2 \sum_{\tau_1} \cdots \sum_{\tau_N} \exp\left(y \sum_{i=1}^{N-1} \tau_i\right)$$

$$= 2 \sum_{\tau_1} e^{y\tau_1} \sum_{\tau_2} e^{y\tau_2} \cdots \sum_{\tau_{N-1}} e^{y\tau_{N-1}}$$

$$= 2\left(e^{y} + e^{-y}\right)^{N-1}, \text{ independence of } \tau_i\text{'s}$$

$$= 2\left[2\cosh(y)\right]^{N-1}$$

$$Z = 2\left[2\cosh(\beta J)\right]^{N-1}$$

c) $U = -\frac{\partial \ln Z}{\partial \beta}$

$= -\frac{\partial}{\partial \beta} \ln\left[2[2\cosh(\beta J)]^{N-1}\right]$

$= -(N-1)\frac{\partial}{\partial \beta} \ln[\cosh(\beta J)]$

$U = -(N-1) J \tanh(\beta J)$ #

For $T \to \infty, \beta \to 0$:

$U \simeq -(N-1)\beta J^2 \to 0$ #

For $T \to 0, \beta \to \infty$, $\tanh(\beta J) \to 1$:

$U \simeq -(N-1)J$ #

d) $\frac{S}{k} = \beta U + \ln Z$

$= -(N-1)\beta J \tanh(\beta J) + (N-1)\ln[2\cosh(\beta J)]$
$+ \ln 2$

$= -(N-1)\beta J \tanh(\beta J) + (N-1)\ln[e^{\beta J}(1+e^{-2\beta J})]$
$+ \ln 2$

$= -(N-1)\beta J \tanh(\beta J) + (N-1)[\beta J + \ln(1+e^{-2\beta J})]$
$+ \ln 2$

$$\frac{S}{k} = (N-1)\left[\beta J + \ln(1+e^{-2\beta J}) - \beta J \tanh(\beta J)\right] + \ln 2$$

$$= (N-1)\left[\beta J(1-\tanh\beta J) + \ln(1+e^{-2\beta J})\right] + \ln 2 \quad \#$$

For $T \to \infty$, $\beta \to 0$

$$\frac{S}{k} \simeq (N-1)\left[\beta J(1-\beta J) + \ln 2\right] + \cancel{\ln} 2$$

$$\to (N-1)\ln 2 + \ln 2 = N \ln 2 \quad \#$$

, make sense since the population of $\sigma_i = \pm 1$ are equal

For $T \to 0$, $\beta \to \infty$

$$\frac{S}{k} \simeq \ln 2 \quad \#$$

, only the ground state contributes which is doubly degenerate