1)

a) The number of states w/ wave number k_x between k_x & $k_x + dk_x$ is

$$dn_x = \frac{L_x}{2\pi} dk_x$$

Also, $$dn_y = \frac{L_y}{2\pi} dk_y$$

Thus the number of states w/ two-dimensional wave vector $\vec{k}$ between $\vec{k} + d\vec{k}$ is:

$$dN \equiv D(\vec{k})\, d^2k = \frac{L_x L_y}{(2\pi)^2} d^2k = \frac{A}{(2\pi)^2} d^2k = \frac{A}{(2\pi\hbar)^2} d^2p$$

where A is the spatial area.

Note that:

$$d^2p = 2\pi p\, dp$$

$$dN = \frac{A}{(2\pi\hbar)^2} 2\pi p\, dp = \frac{A}{(2\pi\hbar)^2} 2\pi p(E) \left(\frac{dp}{dE}\right) dE$$

$$, \quad p = \sqrt{2mE}$$

$$dN = \frac{A}{(2\pi\hbar)^2} 2\pi \sqrt{2mE} \left(\frac{m}{\sqrt{2mE}}\right) dE$$

$$= \frac{Am}{2\pi\hbar^2} dE = D(E)\, dE$$

$$\Rightarrow D(E) = \frac{Am}{2\pi\hbar^2}$$, independent of E

b) $\phi(E) = \dfrac{V}{(2\pi\hbar)^3}\left(\frac{4}{3}\pi p^3\right)$

$= \dfrac{V}{(2\pi\hbar)^3}\,\frac{4}{3}\pi\,(2mE)^{3/2}$

$= \dfrac{V}{2\pi^2\hbar^3}\,\dfrac{(2mE)^{3/2}}{3}$

$\dfrac{d\phi}{dE} = \dfrac{V}{2\pi^2\hbar^3}\,\dfrac{3}{2}\,\dfrac{(2mE)^{1/2}}{3}\,2m$

$= \dfrac{Vm}{2\pi^2\hbar^3}\,(2mE)^{1/2}$ #

c) For $E = pc$ (3-D)

$D(E)\,dE = D(p)\dfrac{dp}{dE}\,dE$

$= \dfrac{V}{(2\pi\hbar)^3}\,4\pi p^2\,\dfrac{dp}{dE}\,dE$

$= \dfrac{4\pi V}{(2\pi\hbar)^3}\,4\pi\left(\dfrac{E}{c}\right)^2\dfrac{d}{dE}(E/c)\,dE = \dfrac{V}{2\pi^2\hbar^3}\,\dfrac{E^2}{c^3}\,dE$

$\Rightarrow D(E) = \dfrac{V}{2\pi^2\hbar^3}\,\dfrac{E^2}{c^3}$ #

For $E = (p^2c^2 + m^2c^4)^{1/2} \Rightarrow p^2 = (E^2 - m^2c^4)/c^2$

$D(E)\,dE = \dfrac{4\pi V}{(2\pi\hbar)^3}\left(\dfrac{E^2 - m^2c^4}{c^2}\right)\dfrac{d}{dE}\left(\dfrac{\sqrt{E^2 - m^2c^4}}{c}\right)dE$

$= \dfrac{V}{2\pi^2\hbar^3}\,\dfrac{E(E^2 - m^2c^4)^{1/2}}{c^3}\,dE$

$\Rightarrow D(E) = \dfrac{V}{2\pi^2\hbar^3}\,\dfrac{E(E^2 - m^2c^4)^{1/2}}{c^3}$ #

Note that only the momentum component in the direction of motion is transformed via Lorentz transformation. Thus we only need to consider the axis that is parallel to the motion:

$$\frac{dp_{\text{parallel}}}{E} \equiv \frac{dp}{E}$$

$$c\,dp' = \gamma(c\,dp - \beta\,dE)$$

$$E' = \gamma(E - \beta c p)$$

$$\frac{c\,dp'}{E'} = \frac{(c\,dp - \beta\,dE)}{E - \beta c p}$$

$$= \frac{dp\left(c - \beta\frac{dE}{dp}\right)}{E - \beta c p}, \quad \text{but } \frac{dE}{dp} = c^2\frac{p}{E}$$

$$= \frac{dp\left(c - \beta c^2 p/E\right)}{E - \beta c p}$$

$$= \frac{c\,dp}{E}\frac{(1 - \beta c p/E)}{(1 - \beta c p/E)}$$

$$\Rightarrow \frac{dp'}{E'} = \frac{dp}{E}$$

we see that $\frac{dp}{E}$ is invariant under Lorentz transformation!