4) $X = \dfrac{m_1 X_1 + m_2 X_2}{m_1+m_2}$, $P = P_1 + P_2$

$\tilde{X} = X_1 - X_2$, $\tilde{P} = \dfrac{m_2 P_1 - m_1 P_2}{m_1+m_2}$ } (1)

a) $[X,P] = \dfrac{1}{m_1+m_2}[m_1 X_1 + m_2 X_2 , P_1 + P_2]$

$= \dfrac{1}{m_1+m_2}\{m_1[X_1,P_1] + m_1[X_1,P_2] + m_2[X_2,P_1] + m_2[X_2,P_2]\}$

$= \dfrac{1}{m_1+m_2}\{m_1(i\hbar) + m_2(i\hbar)\} = i\hbar\dfrac{(m_1+m_2)}{(m_1+m_2)}$

$[X,P] = i\hbar$

$[\tilde{X},\tilde{P}] = \dfrac{1}{m_1+m_2}[X_1 - X_2 , m_2 P_1 - m_1 P_2]$

$= \dfrac{1}{m_1+m_2}\{m_2[X_1,P_1] - m_1[X_1,P_2] - m_2[X_2,P_1] + m_1[X_2,P_2]\}$

$= \dfrac{1}{m_1+m_2}\{m_2(i\hbar) + m_1(i\hbar)\}$

$[\tilde{X},\tilde{P}] = i\hbar$

$[\tilde{X},P] = [X_1 - X_2 , P_1 + P_2] = [X_1,P_1] + [X_1,P_2] - [X_2,P_1] - [X_2,P_2]$

$= i\hbar - i\hbar = 0$

$[X,\tilde{P}] = \dfrac{1}{(m_1+m_2)^2}[m_1 X_1 + m_2 X_2 , m_2 P_1 - m_1 P_2]$

$= \dfrac{1}{(m_1+m_2)^2}\{[X_1,P_1]m_1 m_2 - m_1^2[X_1,P_2] + m_2^2[X_2,P_1] - m_2 m_1[X_2,P_2]\}$

$= \dfrac{1}{(m_1+m_2)^2}\{m_1 m_2 i\hbar - m_1 m_2 i\hbar\} = 0$

b) $H = \dfrac{P_1^2}{2m_1} + \dfrac{P_2^2}{2m_2} + V(X_1 - X_2)$ (2)

from (1):

$\tilde{X} = X_1 - X_2$ (3)

$P = P_1 + P_2$

$(m_1+m_2)\tilde{P} = m_2 P_1 - m_1 P_2$

$\Rightarrow P_1 = \dfrac{m_1 P}{(m_1+m_2)} + \tilde{P}$, $P_2 = \dfrac{m_2 P}{(m_1+m_2)} - \tilde{P}$ (4)

(3) & (4) $\rightarrow$ (2)

$$H = \frac{1}{2m_1}\left(\frac{m_1 P}{(m_1+m_2)} + \tilde{P}\right)^2 + \frac{1}{2m_2}\left(\frac{m_2 P}{m_1+m_2} - \tilde{P}\right)^2 + V(\hat{X})$$

$$= \frac{m_1 P^2}{2(m_1+m_2)^2} + \frac{m_2 P^2}{2(m_1+m_2)^2} + \frac{\tilde{P}^2}{2m_1} + \frac{\tilde{P}^2}{2m_2} + V(\hat{X})$$

$$= \frac{P^2}{2(m_1+m_2)} + \frac{\tilde{P}^2(m_2+m_1)}{2m_1m_2} + V(\hat{X})$$

$$H = \underbrace{\frac{P^2}{2M}}_{\text{describes the motion of the center of mass}} + \underbrace{\frac{\tilde{P}^2}{2\mu} + V(\hat{X})}_{\text{describes the motion of the reduced mass}} \qquad (5)$$

, where

$$\left.\begin{aligned} M &= m_1 + m_2 \\ \mu &= \frac{m_1 \cdot m_2}{(m_1+m_2)} \end{aligned}\right\} \quad (6)$$

In 3D: $$H = \frac{(\vec{P}_1 - \vec{f}_1(\vec{x}_1))^2}{2m_1} + \frac{(\vec{P}_2 - \vec{f}_2(\vec{x}_2))^2}{2m_2} + V(\vec{x}_1 - \vec{x}_2)$$

redefine:

$$\vec{P} = (\vec{P}_1 - \vec{f}_1(\vec{x}_1)) + (\vec{P}_2 - \vec{f}_2(\vec{x}_2))$$

$$\& \ \tilde{\vec{P}} = \frac{1}{(m_1+m_2)}\{m_2(\vec{P}_1 - \vec{f}_1(\vec{x}_1)) - m_1(\vec{P}_2 - \vec{f}_2(\vec{x}_2))\}$$

so we can express H as:

$$H = \frac{\vec{P}^2}{2M} + \frac{\tilde{\vec{P}}^2}{2\mu} + V(\tilde{x})$$

where M & μ are given in (6).