10.7.8

1) $$\left[-\frac{1}{2m}\frac{d^2}{dr^2} + \frac{l(l+1)}{2mr^2} + V(r)\right]u_l(r) = E_l\, u_l(r) \qquad (1)$$

where the radial part $R_l(r) = u_l(r)/r$

(note: $\hbar$ is set to be equal to one)

For $l=0$:

$$\left[-\frac{1}{2m}\frac{d^2}{dr^2} + V(r)\right]u_0(r) = E_0\, u_0(r)$$

$$V(r) = \begin{cases} -V_0 & 0 \le r \le R \\ 0 & R < r \end{cases}$$

We are interested in bound states:

$$E_0 = -|E_0|$$

Thus,

$$\frac{d^2}{dr^2}u_0 + 2m(V_0 - |E_0|)\,u_0 = 0 \quad , 0 \le r \le R$$

$$\frac{d^2 u_0}{dr^2} - 2m|E_0|u_0 = 0 \quad , r > R$$

$$u_0(r) = \begin{cases} A\sin kr + B\cos kr & , 0 \le r \le R \\ Ce^{-Kr} + De^{Kr} & , r > R \end{cases}$$

$$\left.\begin{aligned} k &= \sqrt{2m(V_0 - |E_0|)} \\ K &= \sqrt{2m|E_0|} \end{aligned}\right\} \quad (2)$$

$$R_0(r) = u_0(r)/r$$

$$= \begin{cases} A\dfrac{\sin kr}{r} + B\dfrac{\cos kr}{r}, & 0 \le r \le R \\ C\dfrac{e^{-Kr}}{r} + D\dfrac{e^{Kr}}{r}, & r > R \end{cases}$$

$R_0(r)$ should be finite as $r \to 0$ & $r \to \infty$, thus:

$B = 0$ & $D = 0$

$$R_0(r) = \begin{cases} A\dfrac{\sin kr}{r}, & 0 \le r \le R \\ C\dfrac{e^{-Kr}}{r}, & r > R \end{cases} \quad (3)$$

The wave function & its derivative should be continuous at $r = R$:

$$A\frac{\sin kR}{R} = C\frac{e^{-KR}}{R} \quad (4)$$

& $$Ak\frac{\cos kR}{R} = -KC\frac{e^{-KR}}{R} \quad (5)$$

From (5) & (4):

$$k\cot kR = -K$$

$$\cot kR = -K/k \quad (6)$$

$$-\cot\left(R\sqrt{2m(V_0 - |E_0|)}\right) = \sqrt{\frac{|E_0|}{V_0 - |E_0|}} \quad (7)$$

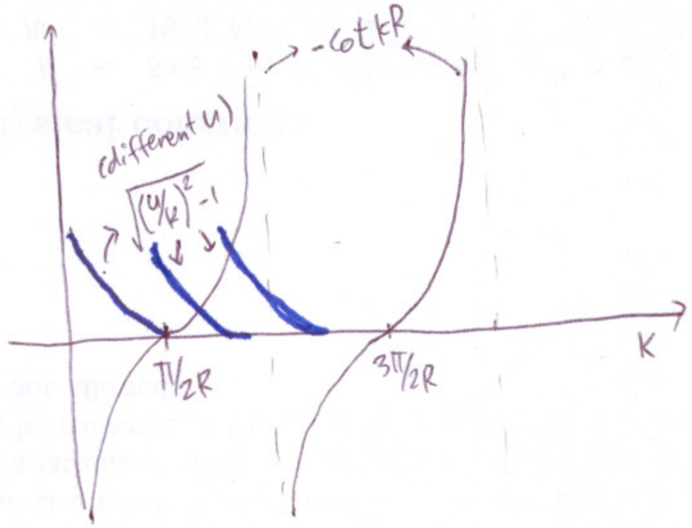
, similar transcendental equation as the odd solution in the 1-D finite square well. The number of solutions depends on the depth & radius of the well. Thus, there cannot be always a bound state.

from (2):

$$\kappa = \sqrt{u^2 - k^2} \quad , \; u^2 \equiv 2mV_0 \qquad (7)$$

(7) → (6):

$$-\cot kR = \sqrt{(u/k)^2 - 1}$$



IF $u = \sqrt{2mV_0} < \dfrac{\pi}{2R}$ there is no bound states (8)

b) $$m = \frac{m_p m_n}{m_p + m_n} = \frac{m_p}{2} \quad , \; m_p \simeq m_n$$

to get only one bound state: (From (8))

$$\sqrt{\frac{2m_p V_0}{\hbar^2}} = \frac{\pi}{2R}$$

note: we put back $\hbar$ here ($\hbar = 1$, previously)

$$V_0 = \frac{\pi^2}{4R^2}\frac{\hbar^2}{2m_p} \quad , \; m_p \cong 938\ \text{MeV}/c^2$$

$$R \simeq 2\,\text{fm}$$

$$\simeq 35\ \text{MeV} \gg B = 2.2\ \text{MeV}$$

c) $V(r) = \frac{A}{r^2} - \frac{B}{r}$, $A \& B > 0$

For $l=0$:

$$\left(-\frac{1}{2m}\frac{d^2}{dr^2} + \frac{A}{r^2} - \frac{B}{r}\right)u_0(r) = E_0 u_0(r)$$

$$\left(\frac{d^2}{dr^2} - \frac{A'}{r^2} + \frac{B'}{r}\right)u_0(r) = E_0' u_0(r)$$

$A' = 2mA$, $B' = 2mB$

$E_0' = -2mE_0 > 0$ ($E_0 < 0$, bound state)

$r \to \infty$:

$$\frac{d^2}{dr^2}u_0(r) = E_0' u_0(r)$$

$u_0(r) \sim \exp(-\sqrt{E_0'}\,r)$, neglected $\exp(\sqrt{E_0'}\,r)$ since it diverges ar $r \to \infty$

$r \to 0$:

$$\frac{d^2 u_0}{dr^2} - \frac{2mA u_0}{r^2} \simeq 0$$

$u_0(r) \sim r^q$, $q = \frac{1+\sqrt{1+4A}}{2} > 0$

neglected $r^{q'}$, $q' = \frac{1-\sqrt{1+4A}}{2}$ since it diverges as r approaches zero

Make an ansatz:

$$u_0(r) = f(r)e^{-\alpha r}, \quad \alpha^2 = \varepsilon'$$

$$\Rightarrow \frac{d^2 f(r)}{dr^2} - 2\alpha\frac{df(r)}{dr} + \frac{B'}{r}f(r) - \frac{A'}{r^2}f(r) = 0 \qquad (a)$$

assume:

$f(r) = \sum_{k=1}^{\infty} a_k r^k$ (this form approaches zero as r approaches zero)

$\rightarrow$ (a):

$$\sum_{k=1}^{\infty} a_k k(k-1)r^{k-2} - 2\alpha\sum_{k=1}^{\infty} a_k k r^{k-1} + B'\sum_{k=1}^{\infty} a_k r^{k-1} - A'\sum_{k=1}^{\infty} a_k r^{k-2} = 0$$

$$\sum_{k=2}^{\infty} a_k k(k-1)r^{k-2} - 2\alpha\sum_{k=1}^{\infty} a_k k r^{k-1} + B'\sum_{k=1}^{\infty} a_k r^{k-1} - A'\sum_{k=1}^{\infty} a_k r^{k-2} = 0$$

$$\sum_{k=0}^{\infty}\left(a_{k+2}(k+2)(k+1)r^k - 2\alpha a_{k+1}(k+1)r^k + B'a_{k+1}r^k - A'a_{k+2}r^k\right) - A'a_1 r^{-1} = 0$$

$$\Rightarrow a_1 = 0$$

$$((k+2)(k+1) - A')a_{k+2} + (B' - 2\alpha(k+1))a_{k+1} = 0$$

$$((k+1)k - A')a_{k+1} = (2\alpha k - B')a_k$$

since $a_1 = 0$, $A' = k(k+1)$ or else $f(r) = 0$

with $A' = \ell(\ell+1)$, the potential turns out to be similar to the Hydrogen potential w/ $B = e^2$!

Thus $E_{n0} = -\dfrac{mB^2}{2n^2}$, $n = a, a+1, a+2, \ldots$

, where $a = \dfrac{1}{2} + \dfrac{\sqrt{1+4a}}{2}$

$$= \frac{-mB^2}{2\hbar^2 n^2} \quad (\text{for } \hbar \neq 1)$$