4.7) $[A,B]=1$

$B|b\rangle = \lambda|b\rangle$

$e^{-\tau A}|b\rangle = ?$

$$[B, e^{-\tau A}] = [B, \sum_{n=0}^{\infty} \frac{(-\tau A)^n}{n!}]$$

$$= \sum_{n=0}^{\infty} \frac{(-\tau)^n}{n!} [B, A^n] \quad (1)$$

note that:

$$[B,A^n] = -[A^n, B]$$

$$= -A^{n-1}[A,B] - [A^{n-1}, B]A$$

$$= -A^{n-1} - A^{n-2}[A,B]A - [A^{n-2}, B]A^2$$

$$= -2A^{n-1} - A^{n-3}[A,B]A^2 - [A^{n-3}, B]A^3$$

$$\vdots$$

$$= -(n-1)A^{n-1} - [A^{n-(n-1)}, B]A^{n-1}$$

$$= -nA^{n-1} \quad (2)$$

(2) → (1):

$$[B, e^{-\tau A}] = -\sum_{n=0}^{\infty} \frac{(-\tau)^n}{n!} nA^{n-1} = -\sum_{n=1}^{\infty} \frac{(-\tau)^n}{n!} nA^{n-1}$$

$$= -\sum_{n=1}^{\infty} \frac{(-\tau)^n}{(n-1)!} A^{n-1} = \tau \sum_{n=1}^{\infty} \frac{(-\tau)^{n-1}}{(n-1)!} A^{n-1}$$

$$[B, e^{-\tau A}] = \tau \sum_{n=0}^{\infty} \frac{(-\tau)^n}{n!} A^n = \tau e^{-\tau A} \quad (3)$$

$$[B, e^{-\tau A}]|b\rangle = \tau e^{-\tau A}|b\rangle$$

$$Be^{-\tau A}|b\rangle - e^{-\tau A}B|b\rangle = \tau e^{-\tau A}|b\rangle$$

$$Be^{-\tau A}|b\rangle - e^{-\tau A}\lambda|b\rangle = \tau e^{-\tau A}|b\rangle$$

$$\Rightarrow Be^{-\tau A}|b\rangle = (\tau+\lambda) e^{-\tau A}|b\rangle$$

$$B|X\rangle = \tau + \lambda |X\rangle$$

where $|X\rangle = e^{-\tau A}|b\rangle$