Jackson 6.12

We start with Jackson (6.136):

$$\frac{1}{2}I_i^* V_i = \frac{1}{2}\int_V \vec{J}^* \cdot \vec{E}\, d^3x + 2i\omega \int_V (w_l - w_m)\, d^3x + \oint_{(S-S_i)} \vec{S}\cdot \hat{n}\, da$$

Using $I_i = YV_i$ ($I_i^* = Y^* V_i^*$),

$$\frac{1}{2}Y^* |V_i|^2 = \int_V \vec{J}^* \cdot \vec{E}\, d^3x + 4i\omega \int_V (w_e - w_m)\, d^3x + 2\oint_{(S-S_i)} \vec{S}\cdot \hat{n}\, da.$$

Finally, since $G = Re\, Y^*$ and $B = Im\, Y^*$,

$$G = \frac{1}{|V_i|^2}\left[Re \int_V \vec{J}^* \cdot \vec{E}\, d^3x - 4\omega\, Im \int_V (w_e - w_m)\, d^3x + 2\, Re \oint_{(S-S_i)} \vec{S}\cdot \hat{n}\, da\right]$$

and

$$B = \frac{1}{|V_i|^2}\left[Im \int_V \vec{J}^* \cdot \vec{E}\, d^3x - 4\omega\, Re \int_V (w_e - w_m)\, d^3x + 2\, Im \oint_{(S-S_i)} \vec{S}\cdot \hat{n}\, da\right]$$

At low frequencies, ignore the surface integral, assume Ohm's law holds ($\vec{J} = \sigma\vec{E}$, σ real), and w_e, w_m are real. Thus,

$$G = \frac{1}{|V_i|^2}\int_V \sigma |\vec{E}|^2 d^3x$$

$$B = \frac{4\omega}{|V_i|^2}\int_V (w_e - w_m) d^3x$$

Jackson 6.14

Use phasors for this problem, so that $I = I_0 \cos\omega t = Re\,(I_0 e^{-i\omega t})$ and Maxwell's equations become

$$\begin{aligned} \nabla\cdot\vec{E} &= \rho/\epsilon, \nabla\times\vec{E} = i\omega\vec{B} \\ \nabla\cdot\vec{B} &= 0, \nabla\times\vec{B} = \mu_0\vec{J} - i\omega/c^2\vec{E} \end{aligned}$$

We can rearrange these equations to get the differential equations for the electric and magnetic fields, $(\nabla^2 + k^2)\{\vec{E}, \vec{B}\} = 0$. Additionally, by ignoring edge effects and using the symmetry of the problem, we note that $\vec{E} = E_z(\rho)\hat{z}$ and $\vec{B} = B_\phi(\rho)\hat{\phi}$. We thus solve our differential equation for E_z:

$$(\nabla^2 + k^2)E_z(\rho) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}E_z(\rho)\right) + k^2 E_z(\rho) = 0$$

The solution is given by a Bessel function,

$$E_z(\rho) = AJ_0(k\rho)$$

To find the amplitude, we use the boundary condition $E_z = \sigma/\epsilon_0$. The alternating current I induces a charge $Q = \int I\,dt = iI_0/\omega e^{-i\omega t}$. To lowest order, though, this is a constant. Since the charge distribution is not necessarily constant over the surface, we integrate over the entire plate and set this equal to the total charge:

$$2\pi \int_0^a E_z \rho \, d\rho = -\frac{Q}{\epsilon_0} = -\frac{iI_0}{\omega\epsilon_0}$$

The integral of J_0 is J_1, so,

$$2\pi \frac{a}{k} J_1(ka) A = -\frac{iI_0}{\omega\epsilon_0}$$

which yields

$$A = -\frac{iI_0 k}{2\pi a\omega\epsilon_0 J_1(ka)} = -\frac{iI_0}{2\pi ac\epsilon_0 J_1(ka)}$$

Thus

$$\vec{E} = -\frac{iI_0 J_0(k\rho)}{2\pi ac\epsilon_0 J_1(ka)}\hat{z}.$$

Using similar reasoning,

$$\vec{B} = -\frac{iI_0 J_1(k\rho)}{2\pi ac^2\epsilon_0 J_1(ka)}\hat{\phi}.$$

6.14(a)

To second order in k,

$$\begin{aligned} J_0(k\rho) &\approx 1 - \frac{k^2\rho^2}{4} + ... \\ J_1(ka) &\approx \frac{ka}{2} + ... \end{aligned}$$

Plugging this in,

$$\begin{aligned} \vec{E} &= -\frac{iI_0}{2\pi ac\epsilon_0}\left[\frac{2}{ka}\left(1 + \frac{k^2a^2}{8} + \frac{k^2\rho^2}{4}\right)\right]\hat{z} \\ \vec{B} &= -\frac{iI_0}{2\pi ac^2\epsilon_0}\left[\frac{\rho}{a}\left(1 + \frac{k^2a^2}{8} - \frac{k^2\rho^2}{8}\right)\right]\hat{\phi} \end{aligned}$$

6.14(b)

$$
\begin{aligned}
w_E &= \frac{\epsilon_0}{4}|\vec{E}|^2 \\
\int w_E\, d^3x &= \frac{\epsilon_0}{4}\frac{I_0^2 d}{4\pi^2 a^2 c^2 \epsilon_0^2}\int_0^a \left[\frac{4}{k^2a^2} + ...\right] \\
&= \frac{I_0^2 d}{4\pi a^2 \omega^2 \epsilon_0}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
w_M &= \frac{1}{4\mu_0}|\vec{B}|^2 \\
\int w_B\, d^3x &= \frac{\mu_0}{4\pi}\frac{I_0^2 d}{8}\left(1 + \frac{\omega^2 a^2}{12c^2}\right)
\end{aligned}
$$

6.14(c)

$$
\begin{aligned}
\chi &= \frac{4\omega}{I^2}\int (w_M - w_E)\, d^3x \\
&= \frac{\mu_0 d\omega}{8\pi}\left(1 + \frac{\omega^2 a^2}{12c^2} - \frac{d}{\epsilon_0 \pi \omega a^2}\right) \\
&= \frac{-1}{\omega C} + \omega L
\end{aligned}
$$

Thus

$$
\begin{aligned}
L &= \frac{\mu_0 d}{8\pi} \\
C &= \frac{\pi \epsilon_0 a^2}{d}
\end{aligned}
$$

The resonance is given by $\omega_{res} = 1/\sqrt{LC} \approx 2\sqrt{2}(c/a)$. From this, $ka \approx 2\sqrt{2} = 2.828$. J_0 has a root near 2.405, which is close to the estimate of ka above.

Jackson 7.16

7.16(a)

Using the substitutions $\partial_t \to -i\omega$ and $\nabla \to i\vec{k}$, Maxwell's relevant equations become $i\vec{k} \times \vec{H} = -i\omega\vec{D}$ and $i\vec{k} \times \vec{E} = i\omega\vec{B}$. Using $\vec{B} = \mu_0\vec{H}$, we get

$$
\vec{k} \times (\vec{k} \times \vec{E}) = i\omega(\vec{k} \times \mu_0\vec{H}) = -\omega^2\mu\vec{D}
$$

7.16(b)

Using vector idententies and $\vec{k} = k\hat{n}$, we write the above result as

$$k^2(\hat{n}\cdot\vec{E})\hat{n} - k^2\vec{E} = -\mu_0 \overleftrightarrow{\epsilon}\, \vec{E}$$

We can write this equation component-wise in a basis that is aligned with the princial axes of the medium:

$$\sum_j \left[n_i n_j + \left(\frac{v^2}{v_i^2} - 1\right)\delta_{ij}\right] E_j = 0.$$

We can define two matrices, $\mathbf{N} = n_i n_j$ and $\mathbf{W} = (1/v_i^2 - 1)\delta_{ij}$ and write this as a eigenvalue problem:

$$(\mathbf{N} - v^2\mathbf{W})\vec{E} = 0$$

In order for this to hold, the $det(\mathbf{N} - v^2\mathbf{W}) = 0$. Many lines of algebra later, it is shown that the Fresnel equations correspond to this determinant. Also, since it is quadratic in v, there are two solutions that are in general different.

7.16(c)

$$\vec{D_a}\cdot\vec{D_b} = \sum_i \epsilon_i^2 E_{ai}E_{bi}$$

Using the Fresnel equations,

$$\begin{aligned}
E_i &= \frac{n_i v_i^2(\hat{n}\cdot\vec{E})}{v_i^2 - v^2} \\
D_i &= \frac{n_i(\hat{n}\cdot\vec{E})}{\mu_0(v_i^2 - v^2)} \\
\vec{D_a}\cdot\vec{D_b} &= \frac{(\hat{n}\cdot\vec{E_a})(\hat{n}\cdot\vec{E_b})}{\mu_0^2}\sum_i \frac{n_i^2}{(v_i^2 - v_a^2)(v_i^2 - v_b^2)} \\
&= \frac{(\hat{n}\cdot\vec{E_a})(\hat{n}\cdot\vec{E_b})}{\mu_0^2(v_a^2 - v_b^2)}\sum_i \left[\frac{n_i^2}{v_i^2 - v_a^2)} - \frac{n_i^2}{v_i^2 - v_a^2)}\right] \\
&= 0
\end{aligned}$$

Extra Problem

Start with the general form of the energy density:

$$\begin{aligned} u &= \frac{1}{4}\left[\frac{\partial}{\partial\omega}(\omega\epsilon)|\vec{E}|^2 + \frac{\partial}{\partial\omega}(\omega\mu)|\vec{B}|^2\right] \\ &= \frac{1}{4}\left[\left(\epsilon+\omega\frac{\partial\epsilon}{\partial\omega}\right)|\vec{E}|^2 + \left(\mu+\omega\frac{\partial\mu}{\partial\omega}\right)|\vec{H}|^2\right] \end{aligned}$$

In the case of plane waves, $k = \omega\sqrt{\mu\epsilon}$ and $\vec{H} = \sqrt{\epsilon/\mu}(\vec{k}\times\vec{E})$, so

$$S = \frac{1}{2}|\vec{E}\times\vec{H}^*| = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}|\vec{E}|^2$$

and

$$u = \frac{1}{4}\left[(\epsilon+\omega\frac{\partial\epsilon}{\partial\omega}) + \frac{\epsilon}{\mu}(\mu+\omega\frac{\partial\mu}{\partial\omega})\right]|\vec{E}|^2$$

Calculate the group velocity as

$$\begin{aligned} v_g = \frac{\partial\omega}{\partial k} &= \left(\frac{\partial\mu}{\partial\omega}\right)^{-1} = \left[\sqrt{\mu\epsilon} + \frac{\omega}{2\sqrt{\mu\epsilon}}\left(\mu\frac{\partial\epsilon}{\partial\omega} + \epsilon\frac{\partial\mu}{\partial\omega}\right)\right]^{-1} \\ &= \frac{1}{\sqrt{\mu\epsilon}}\left(1+\frac{\omega}{2\epsilon}\frac{\partial\epsilon}{\partial\omega} + \frac{\omega}{2\mu}\frac{\partial\mu}{\partial\omega}\right)^{-1} \end{aligned}$$

Calculating the requested ratio,

$$\begin{aligned} \frac{|S|}{u} &= \frac{1/2\sqrt{\epsilon/\mu}|\vec{E}|^2}{1/4\big[(\epsilon+\omega\frac{\partial\epsilon}{\partial\omega}) + \frac{\epsilon}{\mu}(\mu+\omega\frac{\partial\mu}{\partial\omega})\big]|\vec{E}|^2} \\ &= \frac{2\sqrt{\epsilon/\mu}}{1/4\big[(2\epsilon+\omega\frac{\partial\epsilon}{\partial\omega}) + \omega\frac{\partial\mu}{\partial\omega}\big]} \\ &= \frac{1}{\sqrt{\mu\epsilon}}\left(1+\frac{\omega}{2\epsilon}\frac{\partial\epsilon}{\partial\omega} + \frac{\omega}{2\mu}\frac{\partial\mu}{\partial\omega}\right)^{-1} \\ &= v_g. \end{aligned}$$

Thus the energy flows through the material at the group velocity. We can simplify further by noting that $\frac{\partial\epsilon}{\partial\omega} = \frac{\partial\epsilon}{\partial k}\frac{\partial k}{\partial\omega} = 0$ and $\frac{\partial\mu}{\partial\omega} = \frac{\partial\mu}{\partial k}\frac{\partial k}{\partial\omega} = 0$ such the group velocity and energy density reduce to the expressions given in the book.