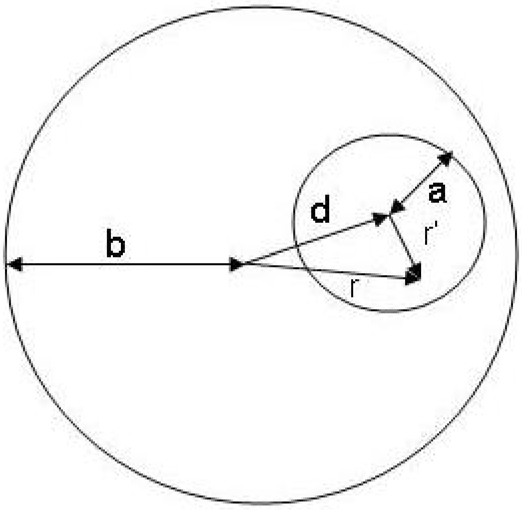

Jackson 5.6

By Ampere's Law, the magnetic field inside a cylindrical conductor carrying current density J is given by

$$\vec{B_{cyl}} = \frac{\mu_0 J(\pi r^2)}{2\pi r}\hat{\theta} = \frac{\mu_0 J r}{2}\hat{\theta}$$

We can consider the geometry in the problem as the superposition of two current-carrying conductors: one with outer radius a carrying current density J, and another of radius b centered at $\vec{d}$ carrying current density $-J$ (see figure). Therefore,

$$\vec{B_{tot}} = \vec{B_{cyl}} - \vec{B_{hole}} = \frac{\mu_0 J r'}{2}\hat{\theta} - \frac{\mu_0 J r'}{2}\hat{\theta}' = \frac{\mu_0 J}{2}(r\hat{\theta} - r'\hat{\theta}').$$

Using $\hat{\theta} = \hat{r} \times \hat{z}$, we get

$$\vec{B_t}ot = \frac{\mu_0 J}{2}[(r\hat{r} \times \hat{z}) - (r'\hat{r'} \times \hat{z})] = \frac{\mu_0 J}{2}[(\vec{r} - \vec{r'}) \times \hat{z}] = \frac{\mu_0 J}{2}(\vec{d} \times \hat{z}).$$

Thus the field inside the hole is constant.

Jackson 5.10

$$\vec{J} = I\delta(z)\delta(\rho - a)\hat{\phi}$$

For parts (a) and (b), keep in mind that

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x'})}{|\vec{x}-\vec{x'}|}\,d^3\vec{x'}$$

We just use different expansions for $|\vec{x}-\vec{x'}|^{-1}$ below.

5.10(a)

Use Eq. (3.143),

$$\frac{1}{|\vec{x}-\vec{x'}|} = \frac{2}{\pi}\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk\, e^{-im(\phi-\phi')}\cos k(z-z')I_m(k\rho_<)K_m(k\rho)$$

We insert this above:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3\vec{x'}\,\vec{J}(\vec{x'})\left[\frac{2}{\pi}\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk\, e^{-im(\phi-\phi')}\cos k(z-z')I_m(k\rho_<)K_m(k\rho)\right] \tag{1}$$

and perform the integrations (see below for details) to get the expression in the book.

5.10(b)

Using the expansion in prob 5.9,

$$\begin{aligned}\frac{1}{|\vec{x}-\vec{x'}|} &= \sum_{m=-\infty}^{\infty}\int_0^{\infty} dk\, e^{im(\phi-\phi')}J_m(k\rho)J_m(k\rho')e^{-k(z_>-z_<)} \\ &= 2\sum_{m=1}^{\infty}\int_0^{\infty} dk\, \cos m(\phi-\phi')J_m(k\rho)J_m(k\rho')e^{-k(z_>-z_<)} + (m=0)\end{aligned}$$

Thus we insert into the expression for $\vec{A}$ (without $m=0$ term) above:

$$\begin{aligned}\vec{A}(\vec{x}) &= \frac{\mu_0}{4\pi}\int d^3\vec{x'}\,\vec{J}(\vec{x'})\left[2\sum_{m=1}^{\infty}\int_0^{\infty} dk\, \cos m(\phi-\phi')J_m(k\rho)J_m(k\rho')e^{-k(z_>-z_<)}\right] \\ &= \frac{\mu_0 I}{2\pi}\sum_{m=1}^{\infty}\int_0^{\infty} dk \int_0^{\infty} d\rho'\, J_m(k\rho)J_m(k\rho')\delta(\rho-a) \\ &\quad \times \int_{-\infty}^{\infty} dz'\, e^{-k|z-z'|}\delta(z')\int_0^{2\pi} d\rho'\, \cos m(\phi-\phi')\hat{\phi}\end{aligned}$$

Considering $\hat{\phi} = \cos\phi\hat{i} + \sin\phi\hat{j}$, we use the orthogonality of trigonometric functions to select only the $m = 1$ term in the sum. Therefore, performing all the integrations, we get the final answer

$$\vec{A}(\vec{x}) = \frac{\mu_0 I a}{2}\int_0^\infty dk J_1(k\rho)J_1(ka)e^{-k|z|}\hat{\phi}$$

5.10(c)

$$B_z = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\phi) = \frac{\partial A_\phi}{\partial\rho} + \frac{1}{\rho}A_\phi$$

Using the expansion in (a),

$$B_z = \frac{\mu_0 I a}{\pi}\int_0^\infty dk\cos(kz)\left[\frac{\partial}{\partial\rho} + \frac{1}{\rho}\right](I_1(k\rho_<)K_1(k\rho_>))$$

On the z-axis, $\rho_< = 0$ and $\rho_> = a$,

$$\begin{aligned} B_z &= \lim_{\rho\to 0}\frac{\mu_0 I a}{\pi}\int_0^\infty dk\cos(kz)\left[\frac{\partial}{\partial\rho} + \frac{1}{\rho}\right](I_1(k\rho)K_1(ka)) \\ &= \frac{\mu_0 I a}{\pi}\int_0^\infty dk\cos(kz)K_1(ka) \\ &= \frac{\mu_0 I a^2}{2(a^2+z^2)^{3/2}} \end{aligned}$$

Jackson 5.19

We use the scalar potential since there is no free current. Additionally, $\rho_m = \nabla\cdot\vec{M} = 0$, so we are only concerned with the surface:

$$\phi_M(z) = \frac{1}{4\pi}\oint_S \frac{\hat{n}\cdot\vec{M}}{|\vec{x}-\vec{x'}|}dS.$$

$\hat{n}\cdot\vec{M} = \pm M_0$ on the top and bottom of the cylinder, respectively, and zero elsewhere. Considering the bottom rests on $z = 0$ and the top is at $z = L$, we perform the integral on the caps:

$$\begin{aligned} \phi_M(z) &= \frac{1}{4\pi}\int_0^a \frac{M_0}{\sqrt{\rho^2+(z-L)^2}} - \frac{M_0}{\sqrt{\rho^2+z^2}}2\pi\rho\,d\rho \\ &= \frac{M_0}{2}\left[\sqrt{a^2+(z-L)^2} - \sqrt{a^2+z^2} - |z| - |z-L|\right] \end{aligned}$$

$$\begin{aligned}\vec{H} &= -\nabla\phi_M = -\frac{\partial}{\partial z}\phi_M(z)\hat{z} \\ &= \begin{cases} \frac{-M_0}{2}\left[\frac{z-L}{\sqrt{a^2+(z-L)^2}} - \frac{z}{\sqrt{a^2+z^2}}\right]\hat{z} & \text{z inside cylinder} \\ \frac{-M_0}{2}\left[\frac{z-L}{\sqrt{a^2+(z-L)^2}} - \frac{z}{\sqrt{a^2+z^2}} - 2\right]\hat{z} & \text{z outside cylinder} \end{cases}\end{aligned}$$

$$\vec{B} = \mu_0(\vec{H}+\vec{M}) = \frac{M_0\mu_0}{2}\left[-\frac{z-L}{\sqrt{a^2+(z-L)^2}} + \frac{z}{\sqrt{a^2+z^2}}\right]\hat{z}$$

for all z.

Jackson 5.29

For a perfect conductor, both $\vec{E}$ and $\vec{B}$ vanish in the interior. Thus we have only surface charges and densities, and the associated potentials (Φ and A_z) are constant in the interiors. (We note that there is only an A_z component because there is no transverse current on the conductor's surface.) The potentials satisfy Poisson's equation on the surface:

$$\begin{aligned}\nabla^2\Phi &= -\rho/\epsilon \\ \nabla^2 A_z &= -K_z\mu\end{aligned}$$

and Laplace's equation on the interior and the interstices. We then calculate the electro- and magnetostatic energy stored in the fields (per unit length):

$$W_E = 1/2\int \rho\phi\, d^3x = Q(V_1 - V_2)/2 = Q^2/2C$$

$$W_M = 1/2\int \vec{J}\cdot\vec{A}\, d^3x = I(A_1 - A_2)/2 = LI^2/2$$

where the V's and A's above are the constant values of Φ and A_z on surfaces 1 and 2. Since A_z and Φ satisfy the same differential equation with the same boundary conditions up to a constant,

$$\frac{A_1 - A_2}{\mu I/a} = \frac{V_1 - V_2}{Q/a\epsilon}$$

Thus

$$LC = \frac{(A_1 - A_2)/I}{(V_1 - V_2)/Q} = \mu\epsilon.$$