

Jackson 1.15

From (1.11),

$$W = \frac{-\epsilon_0}{2} \int_V \phi \nabla^2 \phi dV.$$

Integrating by parts, we get

$$W = \frac{-\epsilon_0}{2} \left(\oint_A \phi \nabla \phi dA - \int_V |\nabla \phi|^2 dV. \right)$$

where the surface integral is over the bounding surfaces. Use $E = -\nabla \phi$ and Gauss' Law:

$$\begin{aligned} W &= \frac{\epsilon_0}{2} \left(\oint_A \phi E dA + \int_V |\nabla \phi|^2 dV. \right) \\ &= \frac{\epsilon_0}{2} \left(\frac{Q_{enc}}{\epsilon_0} + \int_V |\nabla \phi|^2 dV \right) \\ &= \frac{\epsilon_0}{2} \int_V |\nabla \phi|^2 dV \end{aligned} \tag{1}$$

since there is no enclosed charge. To extremize, we must set $\delta W = 0$ when $\phi' = \phi + \delta \phi$. In this case, $|\nabla \phi'|^2 = \nabla \phi' \cdot \nabla \phi' = \nabla \phi \cdot \nabla \phi + 2\nabla \phi \cdot \nabla(\delta \phi) + O(\delta \phi^2)$:

$$\begin{aligned} \delta W &= \frac{\epsilon_0}{2} \int_V 2\nabla \phi \cdot \nabla(\delta \phi) dV \\ &= \epsilon_0 \nabla \phi \delta \phi|_A - \epsilon_0 \int_V \nabla^2 \phi \delta \phi dV \\ &= -\epsilon_0 \int_V \nabla^2 \phi \delta \phi dV = 0. \end{aligned}$$

In the last step, we use the assumption of equipotential which leads to $\nabla \phi = 0$ and the absence of charge. Thus, the energy is extremized.

Jackson 2.1

Using the method of images, we assume an imaginary opposite charge on the other side of the conducting plane. The potential of this two charge system is given by

$$\phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{|\vec{x} - \vec{d}|} + \frac{-q}{|\vec{x} + \vec{d}|} \right)$$

For the remainder of this problem, we will work in cylindrical coordinates and ignore angular components.

2.1(a)

The surface charge is given by

$$\sigma(\rho) = \epsilon_0 E_z(\rho, z = 0) = \epsilon_0 \left. \frac{\partial \phi}{\partial z} \right|_{z=0}$$

Note that $|\vec{x} \pm \vec{d}|^2 = \rho^2 + (r \pm d)^2$. Plugging and chugging,

$$\begin{aligned} \frac{\partial \phi}{\partial z} &= \frac{q}{4\pi\epsilon_0} \frac{\partial}{\partial z} \left(\frac{1}{\sqrt{(z-d)^2 + \rho^2}} - \frac{1}{\sqrt{(z+d)^2 + \rho^2}} \right) \\ &= \frac{q}{4\pi\epsilon_0} \left(\frac{z-d}{((z-d)^2 + \rho^2)^{3/2}} - \frac{z+d}{((z+d)^2 + \rho^2)^{3/2}} \right) \end{aligned}$$

Evaluated at $z = 0$, we arrive at the answer forthwith:

$$\sigma(r) = \epsilon_0 E_z(r, z = 0) = \frac{-dq}{2\pi(d^2 + \rho^2)^{3/2}}$$

2.1(b)

Using Coulomb's law on the image charge:

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{-qq}{(2d)^2} \hat{z} = \frac{-q^2}{16\pi\epsilon_0 d^2} \hat{z}$$

2.1.(c)

In this part we are to calculate the force on the plate by integrating $\sigma^2/2\epsilon_0$ over the plate. To wit:

$$\begin{aligned} \vec{F} &= \int_0^\infty \frac{\sigma(\rho)^2}{2\epsilon_0} \hat{z} 2\pi\rho d\rho \\ &= \hat{z} \frac{d^2 q^2}{4\pi\epsilon_0} \int_0^\infty \frac{\rho}{(\rho^2 + d^2)^3} d\rho \\ &= \hat{z} \frac{d^2 q^2}{4\pi\epsilon_0} \left. \frac{1}{-4(d^2 + \rho^2)^2} \right|_0^\infty = \frac{q^2}{16\pi\epsilon_0 d^2} \hat{z} \end{aligned}$$

2.1(d)

Note first the generalization of the solution to 2.2(c):

$$F_z(z) = \frac{-q^2}{16\pi\epsilon_0 z^2} \hat{z}$$

From there,

$$W = \int_d^\infty -F_z(z) dz = \frac{q^2}{16\pi\epsilon_0} \frac{-1}{z} \Big|_d^\infty = \frac{q^2}{16\pi\epsilon_0 d}$$

2.1(e)

In this case, we calculate the potential energy by multiplying the potential due to the image charge by the real charge, q :

$$W = \frac{1}{8\pi\epsilon_0} \frac{-qq}{2d} = \frac{q^2}{8\pi\epsilon_0 d^2}.$$

In this case, the potential energy is twice that of the answer above. In general, energy calculations with image charges do not work because moving the original charge moves the image. Thus, above, moving the charge to infinity also moved the image to infinity. The actual differential along the path was therefore $2dr$, not dr as in the integral, since both charges moved.

2.1(f)

Plugging in the numbers, we get 3.6 eV, comparable to the work function for most metals.

Jackson 2.3

There are three image line charges in this system: two of charge per unit length $-\lambda$ to the left and below the original charge, and one of charge per unit length λ diagonally from the original charge.

2.3(a)

The potential for the charge configuration above is just the sum of the individual potentials:

$$\Phi_T(\vec{x}) = \frac{\lambda}{4\pi\epsilon_0} \left(\ln \frac{R^2}{|\vec{x} - \vec{x}_0|^2} - \ln \frac{R^2}{|\vec{x} - \vec{x}_1|^2} - \ln \frac{R^2}{|\vec{x} - \vec{x}_2|^2} + \ln \frac{R^2}{|\vec{x} - \vec{x}_3|^2} \right)$$

where \vec{x}_i are the locations of the charges (and $r_i = |\vec{x} - \vec{x}_i|$). At the boundaries, we notice immediately see that the $\Phi = 0$ due to symmetry: at $x = 0$, $|\vec{x} - \vec{x}_0| = |\vec{x} - \vec{x}_3|$ and $|\vec{x} - \vec{x}_1| = |\vec{x} - \vec{x}_2|$. Similar symmetry holds for $y = 0$.

Along the $y = 0$ plane, the tangential field $E_x = \partial\Phi/\partial x$. We note again that the symmetry of the image charges: $\partial r_0/\partial x = \partial r_1/\partial x$ and $\partial r_2/\partial x = \partial r_3/\partial x$, such that all terms cancel.

2.3(b)

Before beginning this, we note that $\partial r_0/\partial y = -\partial r_1/\partial y$ and $\partial r_2/\partial y = -\partial r_3/\partial y$ on the half-plane $y = 0$.

$$\begin{aligned}\sigma(x) &= -\epsilon_0 E_y(x, 0) = -\epsilon_0 \frac{\partial \Phi(x, y)}{\partial y} \Big|_{y=0} \\ &= \frac{-\lambda}{\pi} \left(\frac{y_0}{(x-x_0)^2 + y_0^2} - \frac{y_0}{(x+x_0)^2 + y_0^2} \right) \\ \sigma/\lambda &= \frac{-1}{\pi} \left(\frac{y_0}{(x-x_0)^2 + y_0^2} - \frac{y_0}{(x+x_0)^2 + y_0^2} \right)\end{aligned}$$

2.3(b)

From a table of integrals,

$$\int_0^\infty \frac{1}{(x \mp x_0)^2 + y_0} dx = \frac{1}{2} \frac{\pi \pm 2 \arctan(x_0/y_0)}{y_0}.$$

Thus,

$$Q_x = \int_0^\infty \sigma(x) dx = -\frac{2}{\pi} \lambda \tan^{-1} \left(\frac{x_0}{y_0} \right),$$

as required. Since the charge per unit length z is independent of z , integrating over an infinite plane yields infinite charge.

Jackson 2.9

2.9(a)

The induced charge density is given by (2.15):

$$\sigma = 3\epsilon_0 E \cos \theta.$$

We know that the pressure acting radially outward on the sphere from last homework is $F/A = \sigma^2/2\epsilon_0$. Thus the total force acting on the right hemisphere is

$$\begin{aligned}F_z &= \oint \frac{\sigma^2}{2\epsilon_0} \hat{r} \cdot d\vec{a} \\ &= \frac{1}{2\epsilon_0} \int_{-\pi/2}^{\pi/2} \int_0^\pi 3\epsilon_0^2 E^2 \cos^2 \theta R \cos \theta d\theta d\phi \\ &= \frac{9}{4} \pi \epsilon_0 R^2 E_0^2\end{aligned}$$

2.9(b)

The induced charge density in this case is given by:

$$\sigma = 3\epsilon_0 E \cos \theta + \frac{Q}{4\pi R^2}.$$

Thus,

$$F/A = \frac{\sigma^2}{2\epsilon_0} = \frac{1}{2\epsilon} \left[(3\epsilon_0 E \cos \theta)^2 + \left(\frac{Q}{4\pi R^2} \right)^2 + \frac{6Q\epsilon_0 E \cos \theta}{4\pi R^2} \right].$$

We proceed as before:

$$\begin{aligned} F &= \oint \frac{\sigma^2}{2\epsilon_0} \hat{r} \cdot d\vec{a} \\ &= \frac{1}{2\epsilon_0} \int_{-\pi/2}^{\pi/2} \int_0^\pi (3\epsilon_0 E \cos \theta)^2 + \left(\frac{\pm Q}{4\pi R^2} \right)^2 + \frac{6Q\epsilon_0 E \cos \theta}{4\pi R^2} R d\theta d\phi \\ &= \frac{9}{4} \pi \epsilon_0 R^2 E_0^2 \pm \frac{QE_0}{2} + \frac{Q^2}{32\pi \epsilon_0 R^2} \end{aligned}$$

Note the \pm above corresponds to the left and right hemispheres. The $QE_0/2$ terms in the expressions above point in the same direction, but since this problem was solved in the direction away from the cut, the two hemispheres have differing signs for that term. Additionally, since that term is in the same direction for both halves, it cancels such that the force on one hemisphere is

$$F = \frac{9}{4} \pi \epsilon_0 R^2 E_0^2 + \frac{Q^2}{32\pi \epsilon_0 R^2}.$$

Jackson 2.15

2.15(a)

Obviously, the given form of the Green's function satisfies the boundary conditions on x .

$$\begin{aligned} \nabla^2 G &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) G \\ &= 2 \sum_{n=1}^{\infty} \left(-n^2 \pi^2 g_n(y, y') \sin(n\pi x) \sin(n\pi x') + \frac{\partial^2 g_n}{\partial y^2} \sin(n\pi x) \sin(n\pi x') \right) \\ &= 2 \sum_{n=1}^{\infty} \left(\frac{\partial^2}{\partial y^2} - n^2 \pi^2 \right) g_n \sin(n\pi x) \sin(n\pi x') \\ &= -4\pi \delta(x - x') \delta(y - y') \end{aligned}$$

Using the completeness relation,

$$-4\pi\delta(x - x')\delta(y - y') = -4\pi\delta(y - y') \left(2 \sum_{n=1}^{\infty} \sin(n\pi x) \sin(n\pi x') \right)$$

Thus we equate the two series and the individual terms in the series, giving the relation in the book.

2.15(b)

$$\left(\frac{\partial^2}{\partial y^2} - n^2\pi^2 \right) g_n = -4\pi\delta(y - y')$$

When $y \neq y'$,

$$g_n(y, y') = A_n \sinh(n\pi y_{<}) \sinh(n\pi(1 - y_{>})).$$

Applying the jump condition in first derivative,

$$g'_n(y, y')|_{y'>y} - g'_n(y, y')|_{y>y'} = -4\pi,$$

yields

$$A_n = \frac{4}{n \sinh(n\pi)}.$$

Problem II.A

Analyticity implies that there is a single-valued complex derivative of the complex function $f(z)$. In order for the derivative to exist, it must be the same regardless of the direction of differentiation.

$$\begin{aligned} \left. \frac{df}{dz} \right|_{dz=dx} &= \frac{\partial f_R}{\partial x} + i \frac{\partial f_I}{\partial y} \\ \left. \frac{df}{dz} \right|_{dz=idy} &= -i \frac{\partial f_R}{\partial y} + \frac{\partial f_I}{\partial x} \end{aligned}$$

Equating the real and imaginary parts of the derivatives above yields the Cauchy-Riemann equations:

$$\begin{aligned} \frac{\partial f_R}{\partial x} &= \frac{\partial f_I}{\partial y} \\ \frac{\partial f_I}{\partial y} &= -\frac{\partial f_R}{\partial x} \end{aligned}$$

Differentiating f_R , we find that it solves Laplace's equation:

$$\nabla^2 f_R = \frac{\partial^2 f_R}{\partial x^2} + \frac{\partial^2 f_R}{\partial y^2} = \frac{\partial^2 f_I}{\partial x \partial y} - \frac{\partial^2 f_I}{\partial y \partial x} = 0.$$

Similarly, $\nabla^2 f_I = 0$.

When $f(z) = \arcsin(z) = w$,

$$\exp iw = \sin(w) + i \cos(w) = \sqrt{1 - z^2} + iz$$

$$f(z) = u + iv = -i \ln(\sqrt{1 - z^2} + iz) = -i(|\ln(\sqrt{1 - z^2})| + i \arg(\sqrt{1 - z^2} + iz))$$

Thus the real part is $u = \arg(\sqrt{1 - z^2} + iz)$, which corresponds to a dipole.