[redacted]

30/30

2) Jackson 6.5

a) $\vec{P}_{field} \equiv \int d^3x\, \vec{D}\times\vec{B} = \epsilon_0\mu_0 \int d^3x\, \vec{E}\times\vec{H}$

$= -\frac{1}{c^2}\int d^3x\, \vec{\nabla}\phi\times\vec{H}$, for static fields

$= -\frac{1}{c^2}\int d^3x\left[(\vec{\nabla}\times\phi\vec{H}) - \phi(\vec{\nabla}\times\vec{H})\right]$

$= -\frac{1}{c^2}\oint da\, \hat{r}\times(\phi\vec{H}) + \frac{1}{c^2}\int d^3x\, \phi(\vec{\nabla}\times\vec{H})$ (1)

Provided that $\phi\vec{H} \sim \frac{1}{r^{2+\varepsilon}}$ the surface integral vanishes

& for static fields $\vec{\nabla}\times\vec{H} = \vec{J}$

$\Rightarrow \vec{P}_{field} = \frac{1}{c^2}\int d^3x\, \phi\vec{J}$ #

b) For, $\phi \approx \phi(0) - \vec{\nabla}\phi(0)\cdot\vec{x} = \phi_0 - \vec{E}_0\cdot\vec{x}$

$\vec{P}_{field} = \frac{1}{c^2}\int d^3x\, (\phi_0 - \vec{E}_0\cdot\vec{x})\,\vec{J}(\vec{x})$

$= \frac{\phi_0}{c^2}\cancel{\int d^3x\, \vec{J}(\vec{x})}^{\,0} - \frac{1}{c^2}\int \vec{E}_0\cdot\vec{x}\, \vec{J}(\vec{x})\, d^3x$

consider ith component:

$P_i = -\frac{1}{c^2} E_{0j}\int d^3x\, x_j J_i$

$= -\frac{E_{0j}}{c^2}\frac{1}{2}\int d^3x\,(x_j J_i - x_i J_j) = -\frac{E_{0j}}{c^2}\frac{1}{2}\int d^3x\,(-\epsilon_{ijk}\epsilon_{k\ell m} x_\ell J_m)$

$= -\frac{E_{0j}}{c^2}(-\epsilon_{ijk})\left(\frac{1}{2}\int d^3x\, \epsilon_{k\ell m} x_\ell J_m\right) = \frac{E_{0j}\,\epsilon_{ijk}}{c^2} m_k$

$= \frac{1}{c^2}\epsilon_{ijk} E_{0j} m_k$

$\Rightarrow \vec{P} = \frac{1}{c^2}\vec{E}_0\times\vec{m}$ # (2)

c) From multipole expansion:

$$\vec{A}(\vec{x}) = -\frac{\mu_0}{4\pi}\frac{\hat{r}\times\vec{m}}{r^2} + O\left(\frac{1}{r^3}\right)$$

$$\vec{H}(\vec{x}) = \frac{1}{\mu_0}\vec{\nabla}\times\vec{A}$$

$$= \frac{1}{4\pi}\left(\frac{3\hat{r}(\hat{r}\cdot\vec{m}) - \vec{m}}{r^3}\right) + O\left(\frac{1}{r^4}\right) \qquad (3)$$

Now consider the surface term from part (a) for $\phi = \phi_0 - \vec{E}_0\cdot\vec{x}$:

$$-\frac{1}{c^2}\oint da\, \hat{r}\times\phi\vec{H} = -\frac{1}{c^2}\lim_{r\to\infty}\int d\Omega\, r^2\hat{r}\times\vec{H}(\vec{x})(\phi_0 - \vec{E}_0\cdot\vec{x})$$

$$= -\frac{1}{c^2}\lim_{r\to\infty}\int d\Omega\, r^2\hat{r}\times\left\{\frac{1}{4\pi}\left(\frac{3\hat{r}(\hat{r}\cdot\vec{m})-\vec{m}}{r^3}\right) + O\left(\frac{1}{r^4}\right)\right\}(\phi_0 - \vec{E}_0\cdot\vec{x})$$

$$= -\frac{1}{c^2}\lim_{r\to\infty}\int d\Omega\, r^2\hat{r}\times\left\{\frac{1}{4\pi}\left(\frac{3\hat{r}(\hat{r}\cdot\vec{m})-\vec{m}}{r^3}\right) + O\left(\frac{1}{r^4}\right)\right\}(\phi_0 - \vec{E}_0\cdot\hat{r}r)$$

$$= -\frac{1}{c^2}\lim_{r\to\infty}\int d\Omega\, \hat{r}\times\left\{-\frac{(\vec{E}_0\cdot\hat{r})}{4\pi}(3\hat{r}(\hat{r}\cdot\vec{m})-\vec{m}) + O\left(\frac{1}{r}\right)\right\}$$

$$= \frac{1}{4\pi c^2}\int d\Omega\, \hat{r}\times\{3\hat{r}(\hat{r}\cdot\vec{m}) - \vec{m}\}(\vec{E}_0\cdot\hat{r})$$

$$= -\frac{1}{4\pi c^2}\int d\Omega\, \hat{r}\times\vec{m}\,\vec{E}_0\cdot\hat{r}$$

$$= -\frac{1}{4\pi c^2}\epsilon_{ijk}\hat{x}_i\int d\Omega\, r_j m_k E_{0l}\hat{r}_l$$

$$= -\frac{1}{4\pi c^2}\epsilon_{ijk}\hat{x}_i m_k E_{0l}\left(\frac{4\pi}{3}\delta_{jl}\right), \quad \int d\Omega\, r_i r_j = \frac{4\pi}{3}\delta_{ij}$$

$$= -\frac{1}{3c^2}\epsilon_{ijk}\hat{x}_i m_k E_{0j} = -\frac{1}{3c^2}\vec{E}_0\times\vec{m}$$

Thus,

$$\vec{p}_{field} = \frac{1}{c^2}\vec{E}_0 \times m - \frac{1}{3c^2}\vec{E}_0 \times \vec{m}$$

$$= \frac{2}{3c^2}\vec{E}_0 \times \vec{m} \quad , \quad \vec{E}_0 \equiv \vec{E}(0)$$

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