2) Jackson 6.18

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a) $$\vec{A} = \frac{g}{4\pi}\int_L \frac{d\vec{l}' \times (\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

$$= \frac{g}{4\pi}\int_{-\infty}^{0} \frac{\hat{k}\,dz \times (r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k} - z\hat{k})}{((r\cos\theta - z)^2 + (r\sin\theta)^2)^{3/2}}$$

$$= \frac{g}{4\pi}\int_{-\infty}^{0} \frac{r\,dz\sin\theta\cos\phi\,\hat{j} - r\,dz\sin\theta\sin\phi\,\hat{i}}{(z^2 - 2rz\cos\theta + r^2)^{3/2}}$$

$$= \frac{gr\sin\theta}{4\pi}\hat{\phi}\int_{-\infty}^{0}\frac{dz}{(z^2-2rz\cos\theta+r^2)^{3/2}}$$

$$= \frac{gr\sin\theta}{4\pi}\hat{\phi}\left(\frac{\sin\theta}{r^2(1+\cos\theta)}\right)$$

$$= \frac{g}{4\pi r}\tan\left(\frac{\theta}{2}\right)\hat{\phi} \qquad (1)$$

b) $$\vec{B} = \vec{\nabla}\times\vec{A} = \hat{r}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\left[\frac{g}{4\pi r}\tan\frac{\theta}{2}\right]\right) - \hat{\theta}\frac{\partial}{\partial r}\left(r\left[\frac{g}{4\pi r}\tan\frac{\theta}{2}\right]\right)$$

$$= \hat{r}\frac{g}{4\pi r^2}\left(\csc\theta\left(\frac{1}{2}\sec^2\left(\frac{\theta}{2}\right)\sin\theta + \cos\theta\,\cancel{\tan}\frac{\theta}{2}\right)\right)$$

$$= \hat{r}\frac{g}{4\pi r^2}, \quad \theta \neq \pi \qquad (2)$$

At $\theta = \pi$, the delta-function contribution from Dirac String must be included:

$$\vec{B} = \hat{r}\frac{g}{4\pi r^2} + \hat{z}\frac{g}{4\pi}\delta(x)\delta(y)\Theta(-z) \qquad (3)$$

c) For $\theta < \pi/2$:

$$\int \vec{B}\cdot\hat{n}\,da = \int_0^{\theta} d\theta' \int_0^{2\pi} d\phi' \frac{g}{4\pi r^2} r^2 \sin\theta'$$

$$= \frac{g}{2}(1-\cos\theta) \qquad (4)$$

For $\theta > \pi/2$:

$$\int \vec{B}\cdot\hat{n}\,da = \int_{\theta}^{\pi} d\theta' \int_0^{2\pi} d\phi' \left(\frac{-g}{4\pi r^2}\right) r^2 \sin\theta'$$

$$= -\frac{g}{2}(1+\cos\theta) \qquad (5)$$

d) $$\oint \vec{A}\cdot d\vec{l} = \int_0^{2\pi} \frac{g}{4\pi r} \tan\left(\frac{\theta}{2}\right) r\sin\theta\, d\phi$$

$$= \frac{g}{2}(\csc\theta - \cot\theta)\sin\theta$$

$$= \frac{g}{2}(1-\cos\theta) \qquad (6)$$

Comparing (6) to (5) & (4), we see that for $\theta > \pi/2$ it is off by g where as it is the same for $\theta < \pi/2$. The difference can be athributted to the flux due to the Dirac string.