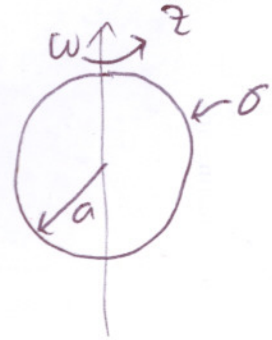
2) 


Jackson 5.13

$$\sigma(r,\theta,\phi) = \sigma\,\delta(r-a)$$

$$\begin{aligned}
\vec{J} &= \sigma(r,\theta,\phi)\,\vec{v} \\
&= \sigma\,\delta(r-a)\,\vec{\omega}\times\vec{r} \\
&= \sigma\,\delta(r-a)\,\omega\hat{z}\times r\hat{r} \\
&= \sigma\omega r\,\delta(r-a)\left[\hat{z}\times(\sin\theta\,\hat{\rho}+\cos\theta\,\hat{z})\right]
\end{aligned}$$

$$\vec{J} = \sigma\omega r\sin\theta\,\delta(r-a)\,\hat{\phi} \qquad (1)$$

For $\vec{J} = J(r,\theta)\,\hat{\phi}$, we can use the relations in Jackson 5.8:

$$\left.\begin{aligned}
A_\phi(r,\theta) &= -\frac{\mu_0}{4\pi}\sum_\ell m_\ell r^\ell P_\ell^1(\cos\theta) \quad , r<a \\
&= -\frac{\mu_0}{4\pi}\sum_\ell \mu_\ell \frac{P_\ell^1(\cos\theta)}{r^{\ell+1}} \quad , r>a
\end{aligned}\right\} \quad (2)$$

where $m_\ell = \dfrac{-1}{\ell(\ell+1)}\displaystyle\int d^3x\,\frac{P_\ell^1(\cos\theta)\,J(r,\theta)}{r^{\ell+1}} \qquad (3)$

$$\mu_\ell = \frac{-1}{\ell(\ell+1)}\int d^3x\, r^\ell P_\ell^1(\cos\theta)\,J(r,\theta) \qquad (4)$$

(1) → (3):

$$m_\ell = \frac{-1}{\ell(\ell+1)}\int_0^{2\pi} d\phi\int_0^\pi \sin\theta\, d\theta\int_0^\infty r^2 dr\,\frac{P_\ell^1(\cos\theta)}{r^{\ell+1}}\, r\omega\sigma\sin\theta\,\delta(r-a)$$

$$= \frac{-2\pi}{\ell(\ell+1)}\,\frac{a^3}{a^{\ell+1}}\,\omega\sigma\int_0^\pi \sin^2\theta\, P_\ell^1(\cos\theta)\, d\theta \qquad (5)$$

(1) → (4)

$$\mu_\ell = \frac{-2\pi}{\ell(\ell+1)}\, a^{\ell+3}\,\omega\sigma\int_0^\pi \sin^2\theta\, P_\ell^1(\cos\theta)\, d\theta \qquad (6)$$

consider:

$$\int_0^\pi \sin^2\theta\, P_\ell^1(\cos\theta)\,d\theta$$

let $x = \cos\theta \Rightarrow x^2 = \cos^2\theta = 1-\sin^2\theta \Rightarrow \sin\theta = (1-x^2)^{1/2}$

$dx = -\sin\theta\, d\theta$

$\Rightarrow$

$$= \int_{-1}^{1} (1-x^2)^{1/2}\, P_\ell^1(x)\,dx$$

$$, \quad P_\ell^1(x) = -(1-x^2)^{1/2} \frac{d}{dx} P_\ell(x)$$

$$= -\int_{-1}^{1} (1-x^2) \frac{dP_\ell(x)}{dx}\, dx$$

$$= -P_\ell(x)(1-x^2)\Big|_{-1}^{1} - \int_{-1}^{1} (+2x) P_\ell(x)\, dx \quad \text{(integration by parts)}$$

$$= -2 \int_{-1}^{1} P_1(x) P_\ell(x)\, dx \; , \quad \text{since } P_1(x) = \cancel{x}$$

$$= -2 \left(\frac{2}{2\ell+1}\right) \delta_{1\ell}$$

$$\Rightarrow \int_0^\pi \sin^2\theta\, P_\ell^1(\cos\theta)\,d\theta = \frac{-4}{2\ell+1}\, \delta_{1\ell} \qquad (7)$$

(7) $\rightarrow$ (5), (6)

$$m_\ell = \frac{+2\pi}{\ell(\ell+1)} \frac{a^3}{a^{\ell+1}}\, \omega\sigma \left(\frac{\cdot 4}{2\ell+1}\right) \delta_{1\ell} \qquad (8)$$

$$M_\ell = \frac{+2\pi}{\ell(\ell+1)}\, a^{\ell+3}\, \omega\sigma \left(\frac{4}{2\ell+1}\right) \delta_{1\ell} \qquad (9)$$

(8), (9) $\rightarrow$ (2)

For $r > a$

$$A_\varphi(r,\theta) = -\frac{\mu_0}{4\pi} \sum_\ell \left( \frac{+2\pi}{\ell(\ell+1)}\, a^{\ell+3}\, \omega\sigma \left(\frac{4}{2\ell+1}\right) \delta_{1\ell} \right) \frac{P_\ell^1(\cos\theta)}{r^{\ell+1}}$$

$$= -\frac{\mu_0}{4\pi}\, \frac{+2\pi}{2}\, a^4 \omega\sigma \left(\frac{4}{3}\right) \frac{P_1^1(\cos\theta)}{r^2}$$

$$A_\varphi(r,\theta) = \frac{\mu_0 \omega \sigma}{3} a^4 \frac{\sin\theta}{r^2} \quad , r > a \qquad (10)$$

note that: $P_1^1(\cos\theta) = \sin\theta$

For $r<a$:

$$A_\varphi(r,\theta) = \frac{-\mu_0}{4\pi} \sum_\ell \left( \frac{2\pi}{\ell(\ell+1)} \frac{a^3}{a^{\ell+1}} \omega\sigma \left(\frac{4}{2\ell+1}\right) \delta_{1\ell} \right) r^\ell P_\ell^1(\cos\theta)$$

$$= \frac{-\mu_0}{4\pi} \left(\frac{2\pi}{2}\right) \frac{a^3}{a^2} \omega\sigma \frac{4}{3} r P_1^1(\cos\theta)$$

$$= \frac{\mu_0 \omega\sigma}{3} a r \sin\theta \quad , r<a \qquad (11)$$

(10) & (11):

$$\vec{A} = \begin{cases} \frac{\mu_0 \omega \sigma}{3} a r \sin\theta \, \hat{\varphi} & , r<a \\ \frac{\mu_0 \omega \sigma}{3} a^4 \frac{\sin\theta}{r^2} \hat{\varphi} & , r>a \end{cases} \qquad (12)$$

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b) $\vec{B} = \vec{\nabla} \times \vec{A}$

$$= \frac{\hat{r}}{r\sin\theta} \left[ \frac{\partial}{\partial\theta} (\sin\theta A_\varphi) \right] - \frac{\hat{\theta}}{r} \frac{\partial}{\partial r} r A_\varphi \quad , \text{since } A_r = A_\theta = 0 \qquad (13)$$

(12) → (13):

$$\vec{B} = \begin{cases} \frac{2\omega\sigma\mu_0 a}{3} [\cos\theta \, \hat{r} - \sin\theta \, \hat{\theta}] & , r<a \\ \frac{\omega\sigma\mu_0 a^4}{3} \left[ \frac{2\cos\theta}{r^3} \hat{r} + \frac{\sin\theta}{r^3} \hat{\theta} \right] & , r>a \end{cases}$$

but $\hat{z} = \cos\theta \, \hat{r} - \sin\theta \, \hat{\theta}$

⇒

$$\vec{B} = \begin{cases} \frac{2\omega\sigma\mu_0 a}{3} \hat{z} & , r<a \\ \frac{\omega\sigma\mu_0 a^4}{3} \left[ \frac{2\cos\theta}{r^3} \hat{r} + \frac{\sin\theta}{r^3} \hat{\theta} \right] & , r>a \end{cases}$$

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