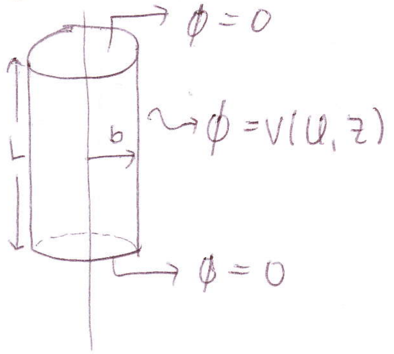
Jackson 3.9



$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

$$\phi = R(\rho)\,Q(\varphi)\,Z(z) \qquad (1)$$

$$\frac{1}{R}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{1}{Q}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{1}{Z}\frac{\partial Z^2}{\partial z^2} = 0$$

choose: $\frac{d^2 Z}{dz^2} + k^2 Z = 0$, $k > 0$

$$\rightarrow Z \sim e^{\pm ikz} \qquad (2)$$

$$\Rightarrow \frac{\rho}{R}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) + \frac{1}{Q}\frac{\partial^2 Q}{\partial\varphi^2} - k^2\rho^2 = 0$$

$$\frac{\partial^2 Q}{\partial\varphi^2} + \nu^2 Q = 0$$

$$\rightarrow Q \sim e^{\pm i\nu\varphi} \qquad (3)$$

$$\Rightarrow \frac{1}{R}\left[\frac{1}{\rho}\frac{\partial R}{\partial\rho} + \frac{\partial^2 R}{\partial\rho^2}\right] - \frac{\nu^2}{\rho^2} - k^2 = 0$$

let $x = k\rho$

$$\frac{d}{d\rho} = \frac{d}{dx}\left(\frac{dx}{d\rho}\right) = k\frac{d}{dx}$$

$$\frac{1}{R}\left[\frac{k}{x}\left(k\frac{dR}{dx}\right) + k^2\frac{d^2R}{dx^2}\right] - \frac{\nu^2k^2}{x^2} - k^2 = 0$$

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} - \left(1+\frac{\nu^2}{x^2}\right) R = 0 \qquad (4)$$

From Jackson page 116, the solutions of (4) are the modified Bessel functions:

$$R(k\rho) = E\, I_\nu(k\rho) + F\, K_\nu(k\rho) \qquad (5)$$

, E & F are constants

$$I_\nu(k\rho) = i^{-\nu} J_\nu(ik\rho)$$

$$K_\nu(k\rho) = \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ik\rho)$$

From (2), (3) & (5):

$$\phi(\rho,\varphi,z) = (A\cos kz + B\sin kz)(C\cos\nu\varphi + D\sin\nu\varphi)$$
$$(E\, I_\nu(k\rho) + F\, K_\nu(k\rho)) \qquad (6)$$

where A, B, C, D, E & F are constants.

Applying boundary conditions & symmetry considerations:

$$\phi(\rho,\varphi,z=0) = 0$$
$$\Rightarrow A = 0 \qquad (7)$$

$$\phi(\rho,\varphi,z=L) = 0$$
$$k = \frac{n\pi}{L}, \quad n = 0,1,2,\ldots \qquad (8)$$

$$\phi(\rho,\varphi+2\pi,z) = \phi(\rho,\varphi,z)$$
$$\Rightarrow \nu = m, \quad m = 0,1,2,\ldots \qquad (9)$$

at $\rho=0$, φ should be finite.

$\Rightarrow F=0$, since $K_\nu(x)$ diverges at $x=0$. (10)

(7), (8), (9) & (10) $\Rightarrow$ (6)

$$\phi = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} I_m\left(\frac{n\pi}{L}\rho\right)\sin\left(\frac{n\pi}{L}z\right)\left\{A_{mn}\sin(m\varphi)+B_{mn}\cos(m\varphi)\right\} \quad (11)$$

, where A_{mn} & B_{mn} are to be determined, yet.

at $\rho=b$:

$$\phi(b,\varphi,z) = V(\varphi,z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} I_m\left(\frac{n\pi}{L}b\right)\sin\left(\frac{n\pi}{L}z\right)\left\{A_{mn}\sin(m\varphi)+B_{mn}\cos(m\varphi)\right\} \quad (12)$$

exploiting the orthogonality of sine & cosine, we'll multiply (12) by $\sin(m'\varphi)/\cos(m'\varphi)$ & $\sin\left(\frac{n'\pi}{L}z\right)$ and then intergrate it over φ & z, to get:

$$\left.\begin{aligned} A_{mn} &= \frac{2}{\pi L\, I_m\left(\frac{n\pi}{L}b\right)}\int_0^L dz\int_0^{2\pi}d\varphi\, V(\varphi,z)\sin(m\varphi)\sin\left(\frac{n\pi}{L}z\right) \\ B_{mn} &= \frac{2}{\pi L\, I_m\left(\frac{n\pi}{L}b\right)}\int_0^L dz\int_0^{2\pi}d\varphi\, V(\varphi,z)\cos(m\varphi)\sin\left(\frac{n\pi}{L}z\right)\end{aligned}\right\} \quad (13)$$

note that, we used the following identities in this step:

$$\int_0^{2\pi}\sin(mx)\sin(nx)\,dx = \int_0^{2\pi}\cos(mx)\cos(nx)\,dx = \pi\,\delta_{mn}$$

$$\int_0^{L}\sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx = \frac{L}{2}\,\delta_{mn}$$

In summary, from (11) & (13):

$$\phi(\rho,\varphi,z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} I_m\left(\frac{n\pi}{L}\rho\right)\sin\left(\frac{n\pi}{L}z\right)\left\{A_{mn}\sin(m\varphi)+B_{mn}\cos(m\varphi)\right\}$$

, where

$$A_{mn} = \frac{2}{\pi L\, I_m\left(\frac{n\pi}{L}b\right)}\int_0^L dz\int_0^{2\pi}d\varphi\, V(\varphi,z)\sin(m\varphi)\sin\left(\frac{n\pi}{L}z\right)$$

$$B_{mn} = \frac{2}{\pi L\, I_m\left(\frac{n\pi}{L}b\right)}\int_0^L dz\int_0^{2\pi}d\varphi\, V(\varphi,z)\cos(m\varphi)\sin\left(\frac{n\pi}{L}z\right)$$