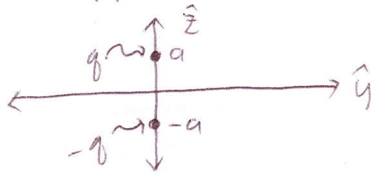
Jackson 3.6



a) $$\phi(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{|\vec{r}-a\hat{z}|} - \frac{1}{|\vec{r}+a\hat{z}|}\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{(r^2+a^2-2ar\cos\theta)^{1/2}} - \frac{1}{(r^2+a^2+2ar\cos\theta)^{1/2}}\right) \qquad (1)$$

for $r>a$:

$$\phi(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r(1+(a/r)^2-2(a/r)\cos\theta)^{1/2}} - \frac{1}{r(1+(a/r)^2+2(a/r)\cos\theta)^{1/2}}\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{a}{r}\right)^n P_n(\cos\theta) - \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{-a}{r}\right)^n P_n(\cos\theta)\right)$$

noting that: $\frac{1}{t^2-2tx+1} = \sum_{n=0}^{\infty} t^n P_n(x)$, $t<1$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r}\right)\sum_{n=0}^{\infty}(1-(-1)^n)\left(\frac{a}{r}\right)^n P_n(\cos\theta)$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{2}{r}\right)\sum_{n=0}^{\infty}\left(\frac{a}{r}\right)^n P_n(\cos\theta), \quad n = \text{odd}$$

$$\phi(\vec{r}) = \frac{q}{2\pi\varepsilon_0}\left(\frac{1}{r}\right)\sum_{n=0}^{\infty}\left(\frac{a}{r}\right)^{2n+1} P_{2n+1}(\cos\theta), \quad r>a \qquad (2)$$

similarly for $r<a$:

$$\phi(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{a((r/a)^2+1-2(r/a)\cos\theta)^{1/2}} - \frac{1}{a((r/a)^2+1+2(r/a)\cos\theta)^{1/2}}\right)$$

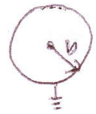
$$= \frac{q}{2\pi\varepsilon_0}\left(\frac{1}{a}\right)\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^{2n+1} P_{2n+1}(\cos\theta), \quad r<a \qquad (3)$$

b) $qa \equiv p/2 = \text{constant as } a \to 0$

, since $a\to 0$ we will use (2)

$$\phi(\vec{r}) = \frac{qa}{2\pi\varepsilon_0 r^2}\sum_{n=0}^{\infty}\left(\frac{a}{r}\right)^{2n} P_{2n+1}(\cos\theta) = \frac{qa}{2\pi\varepsilon_0 r^2}\left(P_1(\cos\theta) + O(a^2)\right)$$

$$\simeq \frac{qa}{2\pi\varepsilon_0 r^2} P_1(\cos\theta) = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}, \quad \text{as } a\to 0 \qquad (4)$$

c) 

at $r<b$:

$$\phi_{shell} = \sum_{n=0}^{\infty} \left(\frac{a_n}{r^{n+1}} + b_n r^n \right) P_n(\cos\theta)$$

$$\phi_{shell}(0) = \text{finite} \rightarrow a_n = 0$$

$$\phi = \phi_{dipole} + \phi_{shell}$$

$$= \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} + \sum_{n=0}^{\infty} b_n r^n P_n(\cos\theta) \qquad (5)$$

at $r=b$

$$\phi(b) = 0$$

$$\frac{p\cos\theta}{4\pi\varepsilon_0 b^2} + b_0 + b_1 b\cos\theta + \sum_{n=2}^{\infty} b_n r^n P_n(\cos\theta) = 0$$

$$b_0 + \left(\frac{p}{4\pi\varepsilon_0 b^2} + b_1 b \right)\cos\theta + \sum_{n=2}^{\infty} b_n r^n P_n(\cos\theta) = 0$$

$P_n(\cos\theta)$ are linearly independent functions thus;

$$b_n = 0, \quad n \neq 1$$

$$b_1 b + \frac{p}{4\pi\varepsilon_0 b^2} = 0$$

$$b_1 = \frac{-p}{4\pi\varepsilon_0 b^3} \qquad (6)$$

(6) → (5):

$$\phi = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} - \frac{p}{4\pi\varepsilon_0 b^3} r\cos\theta$$

$$= \frac{p}{4\pi\varepsilon_0} \left(\frac{1}{r^2} - \frac{r}{b^3} \right)\cos\theta$$

#