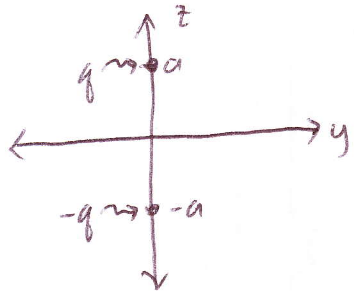
2) Jackson 3.6

8/8



a) $$\phi(\vec{x}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\vec{x}-a\hat{z}|} - \frac{1}{|\vec{x}+a\hat{z}|}\right) \qquad (1)$$

note that:

$$\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta',\varphi') Y_{lm}(\theta,\varphi) \qquad (2)$$

$$\rightarrow \frac{1}{|\vec{x}-a\hat{z}|} = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(0,\varphi') Y_{lm}(\theta,\varphi) \qquad (3)$$

$$\frac{1}{|\vec{x}+a\hat{z}|} = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\pi,\varphi') Y_{lm}(\theta,\varphi) \qquad (4)$$

but $$Y_{lm}^*(0,\varphi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos(0))\, e^{-im\varphi}$$

$$= \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\, (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(1)\, e^{-im\varphi}$$

$$= \begin{cases} 0 & \text{if } m \neq 0 \\ \sqrt{\frac{2l+1}{4\pi}}, & m = 0 \end{cases}, \text{ since } P_l(1) = \text{constant} = (1)^l \qquad (5)$$

similarly,

$$Y^*_{lm}(\pi,\varphi) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi\ (l+m)!}}\ (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(-1)\ e^{-im\varphi}$$

$$= \begin{cases} 0 & \text{if } m \neq 0 \\ \sqrt{\frac{2l+1}{4\pi}}\ (-1)^l & \text{if } m = 0 \end{cases} \qquad (6)$$

(5), (6), (3), (4) $\rightarrow$ (1):

$$\Phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} [1-(-1)^l] \sqrt{\frac{4\pi}{2l+1}}\ \frac{r_<^l}{r_>^{l+1}}\ Y_{l0}(\theta,\varphi)$$

$$= \frac{2q}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} \sqrt{\frac{4\pi}{2(2l+1)+1}}\ \frac{r_<^{2l+1}}{r_>^{(2l+1)+1}}\ Y_{(2l+1),0}(\theta,\varphi)$$

$$= \frac{2q}{\sqrt{4\pi}\varepsilon_0} \sum_{l=0}^{\infty} \sqrt{\frac{1}{4l+3}}\ \frac{r_<^{2l+1}}{r_>^{2l+2}} \left(\sqrt{\frac{2(2l+1)+1}{4\pi}}\ P_{2l+1}(\cos\theta)\right)$$

$$\Phi(\vec{x}) = \frac{q}{2\pi\varepsilon_0} \sum_{l=0}^{\infty} \frac{r_<^{2l+1}}{r_>^{2l+2}}\ P_{2l+1}(\cos\theta) \qquad (6)$$

#

b) $qa \equiv p/2 = \text{constant}$ as $a \rightarrow 0$

since $a \rightarrow 0$, $r_< = a$ & $r_> = r$:

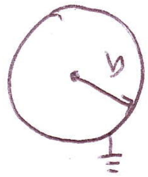
$$\Phi(\vec{x}) = \frac{q}{2\pi\varepsilon_0} \sum_{l=0}^{\infty} \frac{a^{2l+1}}{r^{2l+2}}\ P_{2l+1}(\cos\theta)$$

$$= \frac{qa}{2\pi\varepsilon_0 r^2} \sum_{l=0}^{\infty} \left(\frac{a}{r}\right)^{2l} P_{2l+1}(\cos\theta)$$

$$\phi(\vec{x}) = \frac{qa}{2\pi\varepsilon_0 r^2}\left(P_1(\cos\theta) + O(a^2)\right)$$

$$\approx \frac{qa}{2\pi\varepsilon_0 r^2} P_1(\cos\theta) \quad , \text{as } a \to 0$$

$$\phi(\vec{x}) = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} \qquad (7)$$

c) 

at $r<b$:

$$\phi_{shell} = \sum_{n=0}^{\infty}\left(\frac{a_n}{r^{n+1}} + b_n r^n\right) P_n(\cos\theta)$$

$$\phi_{shell}(0) = \text{Finite}$$

$$\rightarrow a_n = 0$$

$$\phi = \phi_{dipole} + \phi_{shell}$$

$$= \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} + \sum_{n=0}^{\infty} b_n r^n P_n(\cos\theta) \qquad (8)$$

at $r=b$:

$$\phi(b) = 0$$

$$\frac{p\cos\theta}{4\pi\varepsilon_0 b^2} + b_0 + b_1 b\cos\theta + \sum_{n=2}^{\infty} b_n r^n P_n(\cos\theta) = 0$$

$$b_0 + \left(\frac{p}{4\pi\varepsilon_0 b^2} + b_1 b\right)\cos\theta + \sum_{n=2}^{\infty} b_n r^n P_n(\cos\theta) = 0$$

$P_n(\cos\theta)$ are linearly independent functions thus;

$$b_n = 0 \, , \quad n \neq 1$$

$$b_1 b + \frac{p}{4\pi\varepsilon_0 b^2} = 0$$

$$b_1 = \frac{-p}{4\pi\varepsilon_0 b^3} \qquad (9)$$

(9) → (8):

$$\phi = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} - \frac{p}{4\pi\varepsilon_0 b^3} r\cos\theta$$

$$= \frac{p}{4\pi\varepsilon_0}\left(\frac{1}{r^2} - \frac{r}{b^3}\right)\cos\theta \quad \#$$