Jackson 3.13

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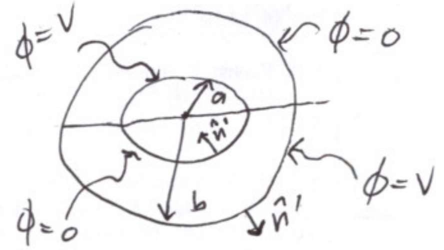

$$G(\vec{x},\vec{x}') = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{Y_{\ell m}^*(\theta',\phi') Y_{\ell m}(\theta,\phi)}{(2\ell+1)\left[1-\left(\frac{a}{b}\right)^{2\ell+1}\right]} \left(r_<^\ell - \frac{a^{2\ell+1}}{r_<^{\ell+1}}\right)\left(\frac{1}{r_>^{\ell+1}} - \frac{r_>^\ell}{b^{2\ell+1}}\right) \quad (1)$$

Only the m=0 terms contribute because of the system's azimuthal sym.:

$$Y_{\ell 0} = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta)$$

$$G(\vec{x},\vec{x}') = \sum_{\ell=0}^{\infty} \frac{P_\ell(\cos\theta) P_\ell(\cos\theta')}{\left[1-\left(\frac{a}{b}\right)^{2\ell+1}\right]} \left(r_<^\ell - \frac{a^{2\ell+1}}{r_<^{\ell+1}}\right)\left(\frac{1}{r_>^{\ell+1}} - \frac{r_>^\ell}{b^{2\ell+1}}\right) \quad (2)$$

$$\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{x}) G(\vec{x},\vec{x}') d^3x' - \frac{1}{4\pi}\oint_S \phi(\vec{x}') \frac{\partial G}{\partial n'} da'$$

$$= -\frac{1}{4\pi}\oint_S \phi(\vec{x}') \frac{\partial G}{\partial n'} da' \text{, since } \rho(\vec{x}) = 0 \text{ inside}$$

$$= -\frac{1}{4\pi}\left[\int_0^\pi \phi(r'=a)\left(\frac{\partial G}{\partial r'}\right)_{r'=a} \hat{r}'\cdot(-\hat{r}')\,(2\pi a^2 \sin\theta' d\theta') + \int_0^\pi \phi(r'=b)\left(\frac{\partial G}{\partial r'}\right)_{r'=b} \hat{r}'\cdot\hat{r}'\,(2\pi b^2 \sin\theta' d\theta')\right]$$

but $\phi(a) = \begin{cases} V, & 0 \le \theta \le \pi/2 \\ 0, & \pi/2 \le \theta \le \pi \end{cases}$ & $\phi(b) = \begin{cases} 0, & 0 \le \theta \le \pi/2 \\ V, & \pi/2 \le \theta \le \pi \end{cases}$

$$\Rightarrow \phi(\vec{x}) = \frac{V}{2}\left[\int_0^{\pi/2}\left(\frac{\partial G}{\partial r'}\right)_{r'=a} a^2 \sin\theta' d\theta' - \int_{\pi/2}^{\pi}\left(\frac{\partial G}{\partial r'}\right)_{r'=b} b^2 \sin\theta' d\theta'\right] \quad (3)$$

note that:

for shell $r'=a$, $r_< = r'$ & $r_> = r$ in (2)

For shell $r'=b$, $r_< = r$ & $r_> = r'$ in (2)

Thus,

$$\left.\frac{\partial G}{\partial r'}\right|_{r'=a} = \sum_{l=0}^{\infty} \frac{P_l(\cos\theta)P_l(\cos\theta')}{\left[1-\left(\frac{a}{b}\right)^{2l+1}\right]}\left(la^{l-1}+\frac{(l+1)a^{2l+1}}{a^{l+2}}\right)\left(\frac{1}{r^{l+1}}-\frac{r^l}{b^{2l+1}}\right)$$

$$= \sum_{l=0}^{\infty} \frac{P_l(\cos\theta)P_l(\cos\theta')}{\left[1-(a/b)^{2l+1}\right]}(2l+1)a^{l-1}\left(\frac{1}{r^{l+1}}-\frac{r^l}{b^{2l+1}}\right) \qquad (4)$$

$$\left.\frac{\partial G}{\partial r'}\right|_{r'=b} = \sum_{l=0}^{\infty} \frac{P_l(\cos\theta)P_l(\cos\theta')}{\left[1-(a/b)^{2l+1}\right]}\left(-\frac{(l+1)}{b^{l+2}}-\frac{l(b^{l-1})}{b^{2l+1}}\right)\left(r^l-\frac{a^{2l+1}}{r^{l+1}}\right)$$

$$= -\sum_{l=0}^{\infty} \frac{P_l(\cos\theta)P_l(\cos\theta')}{\left[1-(a/b)^{2l+1}\right]}\frac{(2l+1)}{b^{l+2}}\left(r^l-\frac{a^{2l+1}}{r^{l+1}}\right) \qquad (5)$$

(4) & (5) → (3):

$$\phi(\vec{x}) = \frac{V}{2}\sum_{l=0}^{\infty}\frac{P_l(\cos\theta)(2l+1)}{\left[1-(a/b)^{2l+1}\right]}\left\{a^{l+1}\left(\frac{1}{r^{l+1}}-\frac{r^l}{b^{2l+1}}\right)\int_0^{\pi/2}P_l(\cos\theta')\sin\theta'd\theta'\right.$$

$$\left.+\frac{1}{b^{l}}\left(r^l-\frac{a^{2l+1}}{r^{l+1}}\right)\int_{\pi/2}^{\pi}P_l(\cos\theta')\sin\theta'd\theta'\right\}$$

$$= \frac{V}{2}\sum_{l=0}^{\infty}\frac{P_l(\cos\theta)(2l+1)}{\left[1-(a/b)^{2l+1}\right]}\left\{a^{l+1}\left(\frac{1}{r^{l+1}}-\frac{r^l}{b^{2l+1}}\right)\int_0^1 P_l(x)\,dx\right.$$

$$\left.+\frac{1}{b^l}\left(r^l-\frac{a^{2l+1}}{r^{l+1}}\right)\int_{-1}^{0}P_l(x)\,dx\right\} \qquad (6)$$

Note that:

$$\int_0^1 P_l(x)\,dx = \begin{cases} 1 & , l=0 \\ 0 & , l=\text{even}, l\neq 0 \\ \left(-\frac{1}{2}\right)^{\frac{l-1}{2}}\dfrac{(l-2)!!}{2\left(\frac{l+1}{2}\right)!} & , l=\text{odd}\end{cases} \qquad (7)$$

also,

$$\int_{-1}^{0}P_l(x)\,dx = (-1)^l\int_0^1 P_l(x)\,dx \qquad (8)$$

(8) → (6):

$$\phi(\vec{x}) = \frac{V}{2}\sum_{l=0}^{\infty} \frac{P_l(\cos\theta)(2l+1)}{[1-(a/b)^{2l+1}]} \left\{ a^{l+1}\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right) + \frac{(-1)^l}{b^l}\left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right)\right\} \int_0^1 P_l(x)\,dx$$

$$= \frac{V}{2}\sum_{l=0}^{\infty} \frac{P_l(\cos\theta)(2l+1)}{[1-(a/b)^{2l+1}]} \left\{ \frac{a^{l+1}}{r^{l+1}} - \frac{(-1)^l a^{2l+1}}{r^{l+1} b^l} - \frac{a^{l+1} r^l}{b^{2l+1}} + \frac{(-1)^l r^l}{b^l}\right\} \int_0^1 P_l(x)\,dx$$

$$= \frac{V}{2}\sum_{l=0}^{\infty} \frac{P_l(\cos\theta)(2l+1)}{[1-(a/b)^{2l+1}]} \left\{ \frac{a^{l+1}}{r^{l+1}}\left(1 - (-1)^l\left(\frac{a}{b}\right)^l\right) - \frac{r^l}{b^l}\left(-(-1)^l + \left(\frac{a}{b}\right)^{l+1}\right)\right\} \int_0^1 P_l(x)\,dx$$

(7) →

$$= \frac{V}{2}\frac{1}{(1-a/b)}\left\{-(-1+(a/b))\right\} + \frac{V}{2}\sum_{l=odd} \frac{P_l(\cos\theta)(2l+1)}{[1-(a/b)^{2l+1}]} \left\{ \frac{a^{l+1}}{r^{l+1}}\left(1+\left(\frac{a}{b}\right)^l\right) - \left(\frac{r}{b}\right)^l \left\{1+\left(\frac{a}{b}\right)^{l+1}\right\}\right\} \frac{(-1/2)^{\frac{l-1}{2}}(l-2)!!}{2\left(\frac{l+1}{2}\right)!}$$

$$= \frac{V}{2} + \frac{V}{4}\sum_{l=odd} \frac{P_l(\cos\theta)(2l+1)}{[1-(a/b)^{2l+1}]} \left\{ \left(\frac{a}{r}\right)^{l+1}\left(1+\left(\frac{a}{b}\right)^l\right) - \left(\frac{r}{b}\right)^l\left(1+\left(\frac{a}{b}\right)^{l+1}\right)\right\} \frac{(-1/2)^{\frac{l-1}{2}}(l-2)!!}{\left(\frac{l+1}{2}\right)!}$$

$$\phi(\vec{x}) = \frac{V}{2} + \frac{V}{4}\sum_{l=0}^{\infty} \frac{P_{2l+1}(\cos\theta)(4l+3)}{[1-(a/b)^{4l+3}]} \frac{(-1/2)^l(2l-1)!!}{(l+1)!} \left\{ \left(\frac{a}{r}\right)^{2l+2}\left(1+\left(\frac{a}{b}\right)^{2l+1}\right) - \left(\frac{r}{b}\right)^{2l+1}\left(1+\left(\frac{a}{b}\right)^{2l+2}\right)\right\} \quad (9)$$

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Expanding (9) up to $l=1$:

$$\phi(\vec{x}) = \frac{V}{2}\left\{1 - \frac{3}{2}\left[(a^2+b^2)r - \frac{(ab)^2(a+b)}{r^2}\right]\frac{P_1(\cos\theta)}{(b^3-a^3)} + \frac{7}{8}\left[(b^4+a^4)r^3 - \frac{(ab)^4(a^3+b^3)}{r^4}\right]\frac{P_3(\cos\theta)}{(b^7-a^7)} + \cdots\right\}$$

it agrees w/ direct solution as done in Prob. 3.1