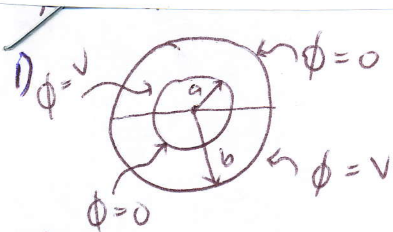
1)



Jackson 3.1

10/10

$$\phi(r,\theta) = \sum_{n=0}^{\infty} \left(A_n r^n + \frac{B_n}{r^{n+1}} \right) P_n(\cos\theta) \qquad (1)$$

Boundary conditions:

$$\phi(a,\theta) = V\, H(\pi/2 - \theta) = \sum_{n=0}^{\infty} \left(A_n a^n + B_n/a^{n+1} \right) P_n(\cos\theta) \qquad (2)$$

step function

$$\phi(b,\theta) = V\, H(\theta - \pi/2) = \sum_{n=0}^{\infty} \left(A_n b^n + B_n/b^{n+1} \right) P_n(\cos\theta) \qquad (3)$$

Multiplying (2) by $P_\ell(\cos\theta)\sin\theta$ & integrating over θ:

$$\int_0^{\pi} P_\ell(\cos\theta)\sin\theta\, V\, H(\pi/2-\theta)\, d\theta = \frac{2}{2\ell+1}\left(A_\ell a^\ell + B_\ell / a^{\ell+1} \right)$$

$$\frac{2V}{2\ell+1} \int_0^{\pi/2} P_\ell(\cos\theta)\sin\theta\, d\theta = A_\ell a^\ell + B_\ell / a^{\ell+1} \qquad (4)$$

Similarly for (3):

$$\int_0^{\pi} P_\ell(\cos\theta)\sin\theta\, V H(\theta-\pi/2)\, d\theta = \frac{2}{2\ell+1}\left(A_\ell b^\ell + B_\ell / b^{\ell+1} \right)$$

$$\frac{2V}{2\ell+1} \int_{\pi/2}^{\pi} P_\ell(\cos\theta)\sin\theta\, d\theta = A_\ell b^\ell + B_\ell / b^{\ell+1} \qquad (5)$$

Evaluating the integrals:

$$\int_0^{\pi/2} P_\ell(\cos\theta)\sin\theta\, d\theta$$

$$= \int_0^1 P_\ell(x)\, dx \,, \quad x = \cos\theta$$

if ℓ is even:

$= \frac{1}{2}\int_{-1}^{1} P_\ell(x) P_0(x)\, dx$, since $P_0(x) = 1$ & $P_\ell(x)$ is even

$= \delta_{\ell 0}$

if l is odd

$$= (-1/2)^{\frac{l-1}{2}} \frac{(l-2)!!}{2\left(\frac{l+1}{2}\right)!}$$ (From an example done in class)

$$\int_{\pi/2}^{\pi} P_l(\cos\theta)\sin\theta \, d\theta = \int_{-1}^{0} P_l(x)\,dx, \quad x = \cos\theta$$

if l is even

$$= \frac{1}{2}\int_{-1}^{1} P_l(x)P_0(x)\,dx$$, since $P_0(x)=1$ & $P_l(x)$ is even

$$= \delta_{l0}$$

if l is odd

$$= -\int_{0}^{1} P_l(x)\,dx$$

$$= -(-1/2)^{\frac{l-1}{2}} \frac{(l-2)!!}{2\left(\frac{l+1}{2}\right)!}$$

$$\Rightarrow \int_{0}^{\pi/2} P_l(\cos\theta)\sin\theta\,d\theta = \begin{cases} \delta_{l0} & \text{, if } l \text{ is even} \\ (-1/2)^{\frac{l-1}{2}} \dfrac{(l-2)!!}{2\left(\frac{l+1}{2}\right)!} & \text{, if } l \text{ is odd} \end{cases} \qquad (6)$$

$$\int_{\pi/2}^{\pi} P_l(\cos\theta)\sin\theta\,d\theta = \begin{cases} \delta_{l0} & \text{, if } l \text{ is even} \\ -(-1/2)^{\frac{l-1}{2}} \dfrac{(l-2)!!}{2\left(\frac{l+1}{2}\right)!} & \text{, if } l \text{ is odd} \end{cases} \qquad (7)$$

From (4), (5), (6) & (7):

$$A_l a^l + B_l/a^{l+1} = \frac{V(2l+1)}{2}\delta_{l0}$$, for l = even

$$A_0 + B_0/a = V/2$$

$$A_l a^l + B_l/a^{l+1} = 0$$, $l = 2, 4, 6, \ldots$

$$A_\ell a^\ell + B_\ell / a^{\ell+1} = V\left(-\tfrac{1}{2}\right)^{\frac{\ell-1}{2}} \frac{(2\ell+1)(\ell-2)!!}{4\left(\frac{\ell+1}{2}\right)!} \quad , \ell = 1, 3, 5, \ldots$$

$$A_0 + B_0/b = V/2$$

$$A_\ell b^\ell + B_\ell / b^{\ell+1} = 0 \quad , \quad \ell = 2, 4, 6, \ldots$$

$$A_\ell b^\ell + B_\ell / b^{\ell+1} = -V\left(-\tfrac{1}{2}\right)^{\frac{\ell-1}{2}} \frac{(2\ell+1)(\ell-2)!!}{4\left(\frac{\ell+1}{2}\right)!} \quad , \ell = 1, 3, 5, \ldots$$

$\Rightarrow$ For $\ell = 0$:

$$A_0 + B_0/a = V/2$$

$$A_0 + B_0/b = V/2$$

$$\Rightarrow (a-b) A_0 = (a-b) V/2$$

$$A_0 = V/2, \quad B_0 = 0 \qquad (8)$$

For $\ell = 1$:

$$A_1 a + B_1/a^2 = \frac{3V}{4}$$

$$A_1 b + B_1/b^2 = -\frac{3V}{4}$$

$$\Rightarrow (a^3 - b^3) A_1 = \frac{3V}{4}(a^2+b^2)$$

$$A_1 = -\frac{3V}{4} \frac{(a^2+b^2)}{(b^3-a^3)}, \quad B_1 = \frac{3V}{4}(ab)^2 \frac{a+b}{b^3-a^3} \qquad (9)$$

For $\ell = 2$:

$$A_2 a^2 + B_2/a^3 = 0$$

$$A_2 b^2 + B_2/b^3 = 0$$

$$(a^5 - b^5) A_2 = 0$$

$$A_2 = 0, \quad B_2 = 0 \qquad (10)$$

For $l=3$:

$$A_3 a^3 + B_3/a^4 = V(-1/2)\,\frac{7}{4}\,\frac{1!!}{(4/2)!} = \frac{-7V}{16}$$

$$A_3 b^3 + B_3/b^4 = \frac{7V}{6}$$

$$\rightarrow (a^7-b^7)A_3 = \frac{-7}{6}V(a^4+b^4)$$

$$A_3 = \frac{7}{16}V\frac{(b^4+a^4)}{(b^7-a^7)},\quad B_3 = -\frac{7}{16}V(ab)^4\frac{a^3+b^3}{b^7-a^7} \qquad (11)$$

For $l-4$:

$$A_4 a^4 + B_4/a^5 = 0$$

$$A_4 b^4 + B_4/b^5 = 0$$

$$(a^4-b^4)A_4 = 0$$

$$A_4 = 0,\ B_4 = 0 \qquad (12)$$

(8), (9), (10), (11), (12) $\rightarrow$ (1):

$$\phi(r,\theta) = \left\{\frac{V}{2} - \frac{3V}{4}\frac{(a^2+b^2)}{(b^3-a^3)}rP_1(\cos\theta) + \frac{3V}{4}(ab)^2\frac{(a+b)}{b^3-a^3}\frac{P_1(\cos\theta)}{r^2}\right.$$

$$\left. + \frac{7}{16}V\frac{(b^4+a^4)}{(b^7-a^7)}r^3P_3(\cos\theta) - \frac{7V}{16}(ab)^4\frac{(a^3+b^3)}{b^7-a^7}\frac{P_3(\cos\theta)}{r^4} + \cdots\right\}$$

good enough!! ✓

$$= \left\{\frac{V}{2} - \frac{3}{4}V\left(\frac{a^2+b^2}{b^3-a^3}r - (ab)^2\frac{(a+b)}{b^3-a^3}\frac{1}{r^2}\right)P_1(\cos\theta)\right.$$

$$\left. + \frac{7}{16}V\left(\frac{a^4+b^4}{b^7-a^7}r^3 - (ab)^4\frac{(a^3+b^3)}{b^7-a^7}\frac{1}{r^4}\right)P_3(\cos\theta) + \cdots\right\}$$

$$\phi(r,\theta) = \frac{V}{2}\left\{1 - \frac{3}{2}\left[(a^2+b^2)r - \frac{(ab)^2(a+b)}{r^2}\right]\frac{P_1(\cos\theta)}{(b^3-a^3)}\right.$$

$$\left. + \frac{7}{8}\left[(b^4+a^4)r^3 - \frac{(ab)^4(a^3+b^3)}{r^4}\right]\frac{P_3(\cos\theta)}{(b^7-a^7)} + \cdots\right\}$$

when $a \to 0$:

$$\phi(r,\theta) \to \frac{V}{2}\left\{1 - \frac{3}{2}\left(\frac{r}{b}\right)P_1(\cos\theta) + \frac{7}{8}\left(\frac{r}{b}\right)^3 P_3(\cos\theta) + \dots\right\}$$

when $b \to 0$:

$$\phi(r,\theta) \cong \frac{V}{2}\left\{1 - \frac{3}{2}\left[b^2 r - \frac{(ab)^2 b}{r^2}\right]\frac{P_1(\cos\theta)}{b^3} + \frac{7}{8}\left[b^4 r^3 - \frac{(ab)^4 b^3}{r^4}\right]\frac{P_3(\cos\theta)}{b^7} + \dots\right\}$$

$$\to \frac{V}{2}\left\{1 + \frac{3}{2}\left(\frac{a}{r}\right)^2 P_1(\cos\theta) - \frac{7}{8}\left(\frac{a}{r}\right)^4 P_3(\cos\theta) + \dots\right\}$$

Very nice