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Jackson 2.19

From Jackson 2.17:

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$$G=\frac{1}{2\pi}\sum_{-\infty}^{+\infty} e^{im(\phi-\phi')}\, g_m(\rho,\rho') \qquad (1)$$

where:

$$\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial g_m}{\partial\rho'}\right)-\frac{m^2}{\rho'^2}g_m=-4\pi\frac{\delta(\rho-\rho')}{\rho}$$

$$\Rightarrow -4\pi\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)-\frac{m^2}{\rho^2}\right]R(\rho)=0 \qquad (2)$$

For $m\neq 0$:

$$\Rightarrow R(\rho)=a_0\rho^m+a_1\rho^{-m}$$

$$\left.\begin{aligned} R_< &= \rho^m-b^{2m}\rho^{-m}\\ R_> &= c^{-2m}\rho^m-\rho^{-m}\end{aligned}\right\} \quad (3)$$

$$W=R_<R_>'-R_<'R_>$$

$$=(\rho^m-b^{2m}\rho^{-m})(mc^{-2m}\rho^{m-1}+m\rho^{-m-1})$$

$$-(m\rho^{m-1}+mb^{2m}\rho^{-m-1})(c^{-2m}\rho^m-\rho^{-m})$$

$$=2m(1-b^{2m}/c^{2m})\,1/\rho$$

$$AW=\frac{1}{p}=\frac{1}{\rho}=A\left[2m\left(1-\left(\frac{b}{c}\right)^{2m}\right)\right]\frac{1}{\rho}\quad , p=\rho/-4\pi$$

$$A=\frac{-4\pi}{2m(1-(b/c)^{2m})} \qquad (4)$$

for $m=0$:

$$R(\rho)=a_0'+a_1'\ln\rho$$

$$\left.\begin{aligned} R_<(\rho) &= -\ln b+\ln\rho\\ R_>(\rho) &= \ln c-\ln\rho\end{aligned}\right\} \quad (5)$$

$$W = (-\ln b + \ln\rho)\left(-\frac{1}{\rho}\right) - (\ln c - \ln\rho)\left(+\frac{1}{\rho}\right)$$

$$= -\left(-\frac{\ln b}{\rho} + \frac{\ln c}{\rho}\right) = \frac{-1}{\rho}\ln(c/b)$$

$$AW = \frac{1}{\rho}, \quad -\frac{A}{\rho}\ln\left(\frac{c}{b}\right) = \frac{-4\pi}{\rho}$$

$$\Rightarrow A = \frac{4\pi}{\ln(c/b)} \qquad (6)$$

From (3) & (4): $(m \neq 0)$

$$g_m = \frac{-4\pi}{2m(1-(b/c)^{2m})}\begin{cases}(\rho^m - b^{2m}\rho^{-m})(c^{-2m}\rho'^m - \rho'^{-m}), & \rho<\rho' \\ (\rho'^m - b^{2m}\rho'^{-m})(c^{-2m}\rho^m - \rho^{-m}), & \rho>\rho'\end{cases}$$

$$= \frac{-4\pi}{2m(1-(b/c)^{2m})}(\rho_<^m - b^{2m}\rho_<^{-m})(c^{-2m}\rho_>^m - \rho_>^{-m})$$

(7), $\rho_< = \rho$ & $\rho_> = \rho'$, $\rho<\rho'$
$\rho_> = \rho$ & $\rho_< = \rho'$, $\rho>\rho'$

From (5) & (6):

$$g_0 = \frac{4\pi}{\ln(c/b)}(-\ln b + \ln\rho_<)(\ln c - \ln\rho_>)$$

$$= \frac{(4\pi)\ln(\rho_</b)\ln(c/\rho_>)}{\ln(c/b)}$$

(8), same notation as (7)

(7), (8) → (1):

$$G(\phi,\rho;\phi',\rho') = \frac{(4\pi)\ln(\rho_</b)\ln(c/\rho_>)}{2\pi\ln(c/b)} + \sum_{m=1}^{\infty}\left(\frac{e^{im(\phi-\phi')} + e^{-im(\phi-\phi')}}{2\pi}\right)$$

$$\frac{-4\pi}{2m(1-(b/c)^{2m})}\left(\rho_<^m - \frac{b^{2m}}{\rho_<^m}\right)\left(\frac{\rho_>^m}{c^{2m}} - \frac{1}{\rho_>^m}\right)$$

$$G(\phi,\rho;\phi',\rho') = \frac{\left[2\ln(\rho_</b)\right]\left[2\ln(c/\rho_>)\right]}{2\ln(c/b)} + 2\sum_{m=1}^{\infty}\frac{\cos[m(\phi-\phi')]}{m(1-(b/c)^{2m})}\left(\rho_<^{m} - \frac{b^{2m}}{\rho_<^{m}}\right)\left(\frac{1}{\rho_>^{m}} - \frac{\rho_>^{m}}{c^{2m}}\right)$$

$$= \frac{\ln(\rho_</b)^2 \ln(c/\rho_>)^2}{\ln(c/b)^2} + 2\sum_{m=1}^{\infty}\frac{\cos[m(\phi-\phi')]}{m(1-(b/c)^{2m})}\left(\rho_<^{m} - \frac{b^{2m}}{\rho_<^{m}}\right)\left(\frac{1}{\rho_>^{m}} - \frac{\rho_>^{m}}{c^{2m}}\right)$$